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A gauge invariant exact renormalization group II

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Abstract: A manifestly gauge invariant and regularized renormalization group flow equation is constructed for pure SU(N) gauge theory in the large-N limit. In this way we make precise and concrete the notion of a non-perturbative gauge invariant continuum wilsonian effective action. Manifestly gauge invariant calculations may be performed, without gauge fixing, and receive a natural interpretation in terms of fluctuating Wilson loops. Regularization is achieved by covariant higher derivatives and by embedding in a spontaneously broken SU($N|N$) supergauge theory; the resulting heavy fermionic vectors are Pauli-Villars fields. We prove the finiteness of this method to one loop and any number of external gauge fields. A duality is uncovered that changes the sign of the squared coupling constant. As a test of the basic formalism we compute the one loop β function, for the first time without any gauge fixing, and prove its universality with respect to cutoff function.

Keywords: Renormalization Regularization and Renormalons, Gauge Symmetry.

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1. Introduction

In ref. [1] we presented a gauge invariant wilsonian RG (renormalization group) [2], formulated directly in the continuum. This formulation was shown to have many attractive features, in particular the fact that manifest gauge invariance can be maintained at all stages of the calculation and thus also in the solution for the effective action S, no gauge fixing or ghosts being required, and the equations may be reinterpreted in terms of fluctuations in the natural gauge invariant order parameters, namely Wilson loops. However, the formulation presented was not sufficient to regularise all ultra-violet divergences. In this paper we solve this problem whilst preserving all these attractive aspects [3].

Our formulation thus furnishes for the first time a precise and concrete realisation of the notion of a non-perturbative gauge invariant continuum wilsonian effective action [3]. In recent years there has been substantial progress in solving supersymmetric gauge theories [4] by computing just such an effective action, even though this object has never been defined. (Only certain general properties were required.) Whilst we concentrate here solely on pure Yang-Mills theory, we see no essential difficulty in generalising the flow equations to include fermions and scalars and indeed spacetime supersymmetry. It is clear then that our framework can underpin these ideas [4, 5].

The regularisation employed in ref. [1] arises essentially from an effective cutoff function [6, 7] which is gauge covariantized. Similarly to gauge covariant higher derivative regularisation, this is not sufficient to regulate all ultra-violet divergences. One loop divergences slip through [8]. In standard perturbation theory, this problem has been cured by supplementing the higher derivative regularisation with a system of Pauli-Villars regulator (PV) fields, the action being bilinear in these fields so that they provide, on integrating out, the missing one loop counterterms[1] [9, 10]. This solution turns out to be unwieldy, but worse, here the property of being bilinear in the PV fields is not preserved by the flow: as the gauge field is integrated out higher-point PV interactions are generated.

Instead, we uncovered a system of regulating fields that is more natural from the exact RG point of view, particularly so in the Wilson loop picture [1], as we first reported in ref. [3]. We have gradually realised that hidden in this formulation are supermatrices and a spontaneously broken local SU($N|N$) (in unitary gauge). We use this insight to give a concise and complete exposition of the formulation sketched in ref. [3].

As in ref. [3], we will concentrate on the gauge group SU(N) in the large-N limit. All the ideas adapt to finite N and indeed other gauge groups, except that the embedding in the appropriate supergauge group should be formulated in such a way as to make this connection more manifest. The disadvantage of the regularisation

[1] And of course other finite contributions.

framework reported here and in ref. [3], is that it was developed intuitively from the bottom up, without us being aware of the underlying local $SU(N|N)$ structure. Whilst many aspects fell out correctly nevertheless, the formulation given in ref. [3] is limited to one loop. Complete regularisation should be achieved in a manifestly local $SU(N|N)$ framework, for reasons that we will outline later. The full exposition of this latter formulation is however left for a future paper.

Such a framework may of course be used independently of the wilsonian RG, and provides a novel and elegant four dimensional 'physical'[2] regularisation for gauge theory which, as we have already intimated, appears to generalise straightforwardly to (chiral) fermions, spacetime supersymmetry and so on. Subtleties in its precise definition and properties [3] are discussed later and in the conclusions, however a full treatment is left to a later paper [11].

One fascinating property reported here,[3] is a duality that effectively exchanges the squared coupling constant g^2 with $-g^2$. At the moment it is not clear to us whether this duality survives in a manifestly local $SU(N|N)$ framework. We comment further in the conclusions.

Let us stress however that there are two main threads in this paper. On the one hand we introduce this natural gauge invariant regularisation, as described above. On the other hand, we go on to use it to repair the divergences reported in ref. [1], and thus develop a consistent calculational framework in which manifest gauge invariance can be maintained at all stages. The fact that the $SU(N)$ gauge invariance is in this way explicitly preserved, thus with no gauge fixing or BRST ghosts, results in elegant and highly constrained relations. One important consequence is that there is no wavefunction renormalization, the only quantity requiring renormalization being the coupling constant [1]. Manifest gauge invariance is also a necessary component of our PV regularisation scheme, as we will show.

Non-perturbative approaches to non-abelian gauge theory that proceed by gauge fixing, must face up to the challenging problem of Gribov copies [12]. Here, these problems are entirely avoided [13]. Indeed we may turn the issue around, and use the present formulation, since it is already well defined without gauge fixing, to investigate explicitly the (quantum) consequences of gauge fixing and Gribov copies directly in the continuum.

In view of the novelty of the construction presented here and in ref. [1], a basic test of the formalism is surely desirable. We compute concrete expressions for all the elementary vertices for two certain choices of covariantization but for general cutoff functions. We then use one of these to derive, directly in the continuum [7], the classical values of the two, three and four point vertices in S and the one-loop contribution to the two-point vertex. From this we compute the one-loop β function.

[2] In the usual sense that it directly suppresses higher momentum modes.

[3] And paranthetically in ref. [3].

Throughout the calculation, we use entirely general cutoff functions,[4] and maintain manifest gauge invariance. The fact that we obtain the result $\beta_1 = -\frac{11}{3}\frac{N}{(4\pi)^2}$, independently of the choice of cutoff functions is encouraging confirmation that the expected universality of the continuum limit has been successfully incorporated. The fact that it agrees with the usual perturbative result, demonstrates for the first time *explicitly* that the one-loop β function is free from Gribov problems, as expected.

The paper is structured as follows. In section 2, after preliminary definitions, we state the flow equation in superfield notation and show that it is manifestly gauge invariant, leaves the partition function invariant, and recall the property of quasilocality and that the flow corresponds to integrating out [1]. This last feature relies on showing that the integrals are indeed ultraviolet regularised, which is established for physical one-loop vertices in section 9 (see below). In setting up the definitions, we also show that the present formulation is a reformulation of that sketched in ref. [3], and show why its form follows essentially from spontaneously broken SU($N|N$). Full and partial supermatrix differentials, the resulting trapped σ_3s, covariantizations, super 'wines', cutoff functions c and $\tilde{c}$, and the renormalization condition are all introduced here. The essence of the calculation can be followed by reading this section, section 3 (which demonstrates that the isolation of perturbative contributions follows in the same way as for the pure gauge case [1]), and sections 7 and 8 that cover respectively the tree-level and one-loop calculation.

However, to arrive at the equations in sections 7 and 8, one needs to extract the Feynman rules from the flow equation. We do this for the reader in section 4. This also serves to fix the nomenclature for the Feynman rules and to explain carefully the precise relation between the flow equation, the Wilson loop diagrams, and their Feynman diagrammatic expansion.

Underlying these rules are the wine vertices themselves. Although these are defined implicitly in section 2, see also refs. [1, 3], clearly we need explicit expressions for concrete calculations. These are derived in section 5, for two forms of covariantization. For the form that we will use here for calculation, we also need to resolve their values at certain special momenta. In this section we also work out the wine vertices' large momentum behaviour. This is not needed for the explicit tree and one-loop calculations, but it is needed at a very rough level to establish finiteness at one loop (again see below). We pause here to work out their large momentum behaviour in much more detail than required, because it is elegant and interesting, falls out with little effort, and may later be important.

In section 6 we uncover and describe the symmetries of the flow equation. On the one hand this helps to understand the equation and the novelties of the underlying SU($N|N$) regularization at a deeper level, in particular we show that charge conjugation invariance and fermion number mix to form a Z_4, that the superfields are only

[4] Up to some basic criteria on normalisation and ultraviolet decay rates.

pseudoreal, and demonstrate the existence of a duality that in a sense exchanges g^2 with $-g^2$. On the other hand, a precise delineation of all symmetries is needed to constrain certain 'counterterms' in the tree-level calculations in section 7.

Indeed in section 7, we will see that the classical vertices suffer a form of divergence as a result of certain freedoms in the Pauli-Villars sector. These in turn lead to the introduction of some new parameters γ. We compute only those vertices that we need for the one-loop calculation in section 8. We also streamline the calculation a little, by borrowing some general results from section 9.

Section 8 starts by showing how the β function is determined, in principle non-perturbatively, following ref. [1]. We then specialize to the concrete one-loop calculation. We explain in particular how we handle the calculation for general cutoff functions by integrating by parts so as to lower the degree of differentiation, whilst bringing all terms to a canonical algebraic form. In this way we reduce the computation to a set of boundary terms in $D = 4$ dimensions which however depend on the power with which the cutoff functions decay. In fact as we already demonstrated in refs. [3, 1], see also [10], the result is ambiguous due to certain total derivative terms that integrate to finite surface terms. Keeping $D \neq 4$ allows all us to discard all such terms, and we find that as $D \to 4$ we recover the famous result for β_1, as already indicated.

In section 9 we give a proof of finiteness for all one-loop physical vertices (i.e. with no external PV fields). We show at the end of the section where the difficulties lie in ensuring finiteness for a larger set of diagrams in this 'unitary gauge' formulation. The first part — up to lemma 3 — explains in broad outline why our exact RG has these finiteness properties; it is these considerations that motivated the form of the exact RG. Although the rest of section 9 stands apart from the paper and can be skipped on first reading, they contain the reasons for some finer details in our flow equation.

Finally in section 10, we present our conclusions, some comparisons with earlier attempts at a manifestly gauge invariant calculations, and indicate future directions.

2. SU($N|N$), Pauli-Villars regularisation, and the exact RG

We formulate the approach in D euclidean dimensions, specializing to $D = 4$ only when required.

In ref. [3] we defined the Pauli-Villars (PV) regulator fields as follows. We took the tensor product of the gauge group with itself, writing $\mathrm{SU}_1(N) \times \mathrm{SU}_2(N)$ to distinguish the two groups. Here we write the hermitean generators of the two groups similarly as τ_1^a and τ_2^a, where each set of generators are orthonormalised as $\mathrm{tr}(\tau^a\tau^b) = \frac{1}{2}\delta^{ab}$. Their associated gauge fields are A^1_μ, the physical gauge field, and A^2_μ, which is unphysical and part of the regularisation scheme. (The fields are valued in the Lie algebra i.e. $A^i_\mu \equiv A^{i\,a}_{\ \mu}\tau^a_i$, $i = 1, 2$.) We introduced [3] a fermionic

Pauli-Villars (PV) field $(B_\mu)^{i_1}_{j_2}$ and its complex conjugate $(\bar{B}_\mu)^{i_2}_{j_1}$. As indicated, B transforms as the fundamental of $\mathrm{SU}_1(N)$ and as the complex conjugate fundamental of $\mathrm{SU}_2(N)$. Finally, we introduced two bosonic real scalar adjoint PV fields C^1 and C^2. The A^i's are massless (of course) while B and the C^i have masses at the effective cutoff Λ. Originally we figured out the statistics, representation content, and the interactions of these fields intuitively in such a way as to ensure finiteness of the quantum corrections in the RG flow equation. We found that there was very little freedom in the choice of interactions if this was to be achieved. The majority of the third lecture in ref. [3] was devoted to describing the construction from this point of view, and will not be repeated here.

We now understand these choices in terms of spontaneously broken $\mathrm{SU}(N|N)$. Let the supergauge field of $\mathrm{SU}(N|N)$ be $\mathcal{A}_\mu$, which we write in supermatrix form, i.e. as a matrix representation with bosonic diagonal elements, and fermionic off-diagonal elements:

$$\mathcal{A}_\mu = \begin{pmatrix} A^1_\mu & B_\mu \\ \bar{B}_\mu & A^2_\mu \end{pmatrix}. \tag{2.1}$$

Since $\mathcal{A}$ is valued in the graded Lie algebra of $\mathrm{SU}(N|N)$, we require that its supertrace vanishes: $\mathrm{str}\,\mathcal{A} = 0$, where the supertrace is defined by

$$\mathrm{str}\,\mathbf{X} = \mathrm{str}\begin{pmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{pmatrix} = \mathrm{tr}\,X^{11} - \mathrm{tr}\,X^{22}$$

for any supermatrix $\mathbf{X}$. (The extra signs incurred on commutation mean that for such matrices it is only the supertrace that is cyclically symmetric, and thus in particular $\mathrm{SU}(N|N)$ invariant.) We also introduce the superscalar

$$\mathcal{C} = \begin{pmatrix} C^1 & D \\ \bar{D} & C^2 \end{pmatrix}, \tag{2.2}$$

but do not require $\mathrm{str}\,\mathcal{C} = 0$.

Note that the conditions $\mathrm{tr}\,C^1 = \mathrm{tr}\,C^2 = 0$ are thus not imposed as in ref. [3]. Similarly, the conditions $\mathrm{tr}\,A^1_\mu = \mathrm{tr}\,A^2_\mu = 0$ are not directly imposed. Only the constraint $\mathrm{tr}\,A^1_\mu - \mathrm{tr}\,A^2_\mu = 0$ has been applied so far. Actually we can, and here will, require that the $\mathrm{U}(1)$ components of the supermatrix algebra, $\mathrm{tr}(A^1_\mu + A^2_\mu)$ and $\mathrm{tr}(C^1 + C^2)$, are also absent, by a suitable modification of the matrix commutator representation of the super-Lie product in $\mathrm{SU}(N|N)$ [14].

(Although it is not required for the rest of the paper, it may help to pause on these points. Bars' observation is that the super-Lie product may be equally well represented by a bilinear (anti)symmetric "$*$" bracket

$$[\ ,\ ]^*_\pm = [\ ,\ ]_\pm - \frac{1}{2N}\mathrm{tr}[\ ,\ ]_\pm\,,$$

where $[\ ,\ ]_\pm$ is applied to the supergenerators and is a commutator or anticommutator as appropriate. This effectively removes the unit matrix as a representative of the algebra, and thus its bosonic U(1) subgroup. In fact we will handle these U(1) components in a different way in our proof that spontaneously broken $SU(N|N)$ acts as a regulator [11]. For the present paper it is only important to note that these modifications are subleading in N and thus vanish in the $N = \infty$ limit that we will take shortly. Let us stress that there are in any case no U(1) components in the present formulation [3]. In this paper, we use $SU(N|N)$ to interpret and systemize our earlier results [3], and in this interpretation we may understand the missing U(1) factors in the way that we have just described.)

We choose the lagrangian so that $\mathcal{C}$ picks up the expectation value $\langle\mathcal{C}\rangle \sim \Lambda\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, breaking $SU(N|N)$ spontaneously to $SU(N) \times SU(N)$. In unitary gauge the Goldstone modes D vanish (eaten by the B_μ's), leaving the massive 'Higgs' C^i and massive vector fermions B_μ. In this way, we elegantly recover exactly the spectrum of fields introduced in ref. [3]. Remarkably, with covariant higher derivative regularisation, we also recover, up to some small details, the $\hat{S}$ interactions used in ref. [3] for the one-loop β function calculation!

Note that the only remaining massless fields are the gauge fields of $SU(N) \times SU(N)$, which are of course charge neutral under each others gauge group. Thus at energies much less than Λ we are left only with these gauge fields which decouple into the required physical $SU(N)$ Yang-Mills theory, and a copy which as we will see has the opposite sign squared coupling constant and is thus unphysical.

Not surprisingly, the $SU(N|N)$ theory described above has very good ultra-violet behaviour. Technically this arises in the unbroken theory, because any quantum correction involving $\mathrm{tr}\,1 = N$ in the $SU(N)$ theory, here involves $\mathrm{str}\,1 = 0$. (We will refer to this as 'the supertrace mechanism'; of course at the level of component fields it arises through exact cancellation between bosonic and fermionic degrees of freedom.) Indeed in the large-N limit the symmetric phase of the supertheory thus has no quantum corrections at all. With covariant higher derivative regularisation, we expect to be able to ensure that the remaining corrections even at $N \neq \infty$ are finite. However, the full development of these investigations is left for the future.

Here, and from now on, we will take the $N = \infty$ limit and work solely with the theory described in ref. [3], which is closely related to spontaneously broken $SU(N|N)$ in unitary gauge as described above (and further outlined below). As we will show, it appears that we cannot ensure the full regularisation of the theory this way except to one loop with external gauge fields. We interpret this as symptoms of the differences described above and of the expected poor ultraviolet behaviour of the unitary gauge.

It is helpful to introduce the $2N \times 2N$ 'Pauli' matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(in terms of which for example we have $\mathrm{str}\,\mathbf{X} = \mathrm{tr}\,\sigma_3\mathbf{X}$) and to separate the bosonic and fermionic parts of the superfields, thus

$$\mathbf{A}_\mu = \begin{pmatrix} A^1_\mu & 0 \\ 0 & A^2_\mu \end{pmatrix}, \quad \mathbf{B}_\mu = \begin{pmatrix} 0 & B_\mu \\ \bar{B}_\mu & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} C^1 & 0 \\ 0 & C^2 \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega^1 & 0 \\ 0 & \omega^2 \end{pmatrix},$$

where ω is the unbroken $\mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge transformation. Defining now

$$\mathbf{D}_\mu = \partial_\mu - i\mathbf{A}_\mu \qquad \text{and} \qquad \nabla_\mu = \partial_\mu - i\mathcal{A}_\mu\,,$$

both $\mathbf{A}$, and $\mathcal{A} = \mathbf{A} + \mathbf{B}$, gauge transform under $\mathrm{SU}(N) \times \mathrm{SU}(N)$:

$$\delta\mathbf{A}_\mu = \mathbf{D}_\mu \cdot \omega := [\mathbf{D}_\mu, \omega]\,, \qquad \delta\mathcal{A}_\mu = \nabla_\mu \cdot \omega \equiv [\nabla_\mu, \omega]\,. \tag{2.3}$$

The formalism of ref. [1] can be lifted to the supertheory as follows. Functional derivatives are defined with respect to the supertrace, thus

$$\begin{aligned} \frac{\delta}{\delta\mathbf{A}_\mu} &= \begin{pmatrix} \delta/\delta A^1_\mu & 0 \\ 0 & -\delta/\delta A^2_\mu \end{pmatrix}, & \frac{\delta}{\delta\mathbf{B}_\mu} &= \begin{pmatrix} 0 & -\delta/\delta\bar{B}_\mu \\ \delta/\delta B_\mu & 0 \end{pmatrix} \\ \frac{\delta}{\delta\mathbf{C}} &= \begin{pmatrix} \delta/\delta C^1 & 0 \\ 0 - \delta/\delta C^2 & \end{pmatrix}, & \frac{\delta}{\delta\mathcal{A}} &= \frac{\delta}{\delta\mathbf{A}} + \frac{\delta}{\delta\mathbf{B}}\,, \end{aligned} \tag{2.4}$$

(where the adjoint derivatives may be defined by $\delta/\delta A^i = 2\tau^a_i\delta/\delta A^{ia}$, $i = 1, 2$, as in ref. [1]. At large N, $\delta/\delta C^i$ will also be thought of as an adjoint derivative, i.e. we take the C^i to be traceless. Again this restriction makes no difference in the large-N limit, but it is convenient for the C^i terms to inherit in this way the conventions associated with the τ^a_i being normalised to 1/2.)

These transform homogeneously; their other important properties are easier to state by ignoring the x dependence and μ index (as in ref. [1]). Let $\mathbf{M}$, $\mathbf{Y}$ and $\mathbf{Z}$ also be supermatrix representations as defined above. If $s(\mathbf{M})$ is a (bosonic) function of $\mathbf{M}$ such that

$$\delta s(\mathbf{M}) = \mathrm{str}\,\delta\mathbf{M}\,\mathbf{Y}\,, \tag{2.5}$$

then, in precise analogy with ref. [1],

$$\frac{\partial s}{\partial\mathbf{M}} = \mathbf{Y}\,. \tag{2.6}$$

(Here in the adjoint parts we use the completeness relation for $\mathrm{SU}(N)$, which leads to the same relation as the unrestricted functional derivative, up to $1/N$ corrections that ensure the coefficient matrix is projected onto its traceless part [1]. These corrections can be neglected since, as stated above, we are only interested from here on, in the $N = \infty$ limit.)

This leads to 'supersowing' under the supertrace

$$\mathrm{str}\,\mathbf{X}\frac{\partial s}{\partial\mathbf{M}} = \mathrm{str}\,\mathbf{X}\mathbf{Y}\,, \tag{2.7}$$

and to 'supersplitting'

$$\delta\mathbf{X} = \mathbf{Y}\delta\mathbf{M}\,\mathbf{Z} \qquad \Longrightarrow \qquad \mathrm{str}\,\frac{\partial}{\partial\mathbf{M}}\mathbf{X} = \mathrm{str}\,\mathbf{Y}\,\mathrm{str}\,\mathbf{Z}\,. \tag{2.8}$$

If however $\mathbf{M}$ is only block diagonal like $\mathbf{A}$ or $\mathbf{C}$, equivalently purely bosonic, then $\mathbf{Y}$ in (2.6) and (2.7) is replaced by $\mathrm{d}_+\mathbf{Y}$, the block diagonal (a.k.a. bosonic) part of $\mathbf{Y}$. On the other hand, if $\mathbf{M}$ is only block off-diagonal, like $\mathbf{B}$, equivalently purely fermionic, then $\mathbf{Y}$ in (2.6)–(2.7) is replaced by $\mathrm{d}_-\mathbf{Y}$, the block off-diagonal (a.k.a. fermionic) part of $\mathbf{Y}$.

The projectors $\mathrm{d}_\pm$ may be expressed as:

$$\mathrm{d}_\pm\mathbf{Y} = \frac{1}{2}\left(\mathbf{Y} \pm \mathbf{Y}^-\right), \tag{2.9}$$

where $\mathbf{Y}^- = \sigma_3\mathbf{Y}\sigma_3$ has opposite sign fermionic components compared to $\mathbf{Y}$. Either from this or directly, we have the identities

$$\mathrm{str}\,\mathbf{X}\mathrm{d}_\pm\mathbf{Y} = \mathrm{str}(\mathrm{d}_\pm\mathbf{X})\mathbf{Y} = \mathrm{str}\,\mathrm{d}_\pm\mathbf{X}\mathrm{d}_\pm\mathbf{Y}\,. \tag{2.10}$$

Summarising, for a 'partial' supermatrix $\mathbf{M} = \mathrm{d}_\pm\mathbf{M}$, (2.7) is altered to

$$\mathrm{str}\,\mathbf{X}\frac{\partial s}{\partial\mathbf{M}} = \mathrm{str}\,\mathbf{X}\mathrm{d}_\pm\mathbf{Y}\,. \tag{2.11}$$

These alterations and the relations (2.10) are obvious in the bosonic/fermionic language.

Similarly, for a partial supermatrix $\mathbf{M} = \mathrm{d}_\pm\mathbf{M}$, the splitting relation (2.8) is altered:

$$\mathrm{str}\,\frac{\partial}{\partial\mathbf{M}}\mathbf{X} = \frac{1}{2}\left(\mathrm{str}\,\mathbf{Y}\,\mathrm{str}\,\mathbf{Z} \pm \mathrm{str}\,\sigma_3\mathbf{Y}\,\mathrm{str}\,\sigma_3\,\mathbf{Z}\right), \tag{2.12}$$

as follows most readily by writing $\delta\mathbf{X} = \mathbf{Y}\mathrm{d}_\pm(\delta\mathbf{M})\,\mathbf{Z}$, after which the 'full' supersplitting relation (2.8) may be used. This 'broken' supersplitting relation (which in a local $SU(N|N)$ formulation would be seen to arise from insertions of $\langle\mathcal{C}\rangle \sim \Lambda\sigma_3$) provides the only reason why there are any quantum corrections at all in the large-N limit. As discussed in ref. [1], the quantum corrections split open the traces, and in the large-N limit, they survive only in the terms where all fields vacate one of the two traces so as to leave $\mathrm{tr}\,1 = N$. In the present case only the broken terms survive, through $\mathrm{str}\,\sigma_3 = 2N$; the unbroken quantum corrections have no way to survive because vacating a supertrace leaves behind $\mathrm{str}\,1$ which vanishes. Expanding (2.12) in $N \times N$ block components, in the bosonic case we have $\mathrm{tr}\,Y^{11}\,\mathrm{tr}\,Z^{11} + \mathrm{tr}\,Y^{22}\,\mathrm{tr}\,Z^{22}$, and in the fermionic case $-\,\mathrm{tr}\,Y^{11}\,\mathrm{tr}\,Z^{22} - \mathrm{tr}\,Y^{22}\,\mathrm{tr}\,Z^{11}$. Note the minus signs in the latter, as expected for a fermionic loop.

Recall that a central construction of the gauge invariant flow equation in ref. [1], is the 'wine', the covariantization of a (smooth) momentum space kernel $W_p \equiv$

$W(p^2/\Lambda^2)$. In position space we write the kernel as

$$W_{xy} \equiv \int \frac{d^D p}{(2\pi)^D} W\left(\frac{p^2}{\Lambda^2}\right) \mathrm{e}^{ip\cdot(x-y)} . \tag{2.13}$$

For two $N \otimes \bar{N}$ representations of the gauge group SU(N), $v(y)$ and $u(x)$, we write

$$\begin{aligned} u\{W\}v = \sum_{m,n=0}^{\infty} \int & d^D x\, d^D y\, d^D x_1 \cdots d^D x_n\, d^D y_1 \cdots d^D y_m \times \\ & \times W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(x_1,\ldots,x_n;y_1,\ldots,y_m;x,y) \times \\ & \times \mathrm{tr}\Big[u(x)\, A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\, v(y)\, A_{\nu_1}(y_1)\cdots A_{\nu_m}(y_m)\Big] , \end{aligned} \tag{2.14}$$

where without loss of generality we may insist that $\{W\}$ satisfies $u\{W\}v \equiv v\{W\}u$. This equation defines the wine vertices $W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}$, given the method of covariantization [1]. As in [1], we write the $m = 0$ vertices (where there is no second product of gauge fields), more compactly as

$$W_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n;x,y) \equiv W_{\mu_1\cdots\mu_n,}(x_1,\ldots,x_n;;x,y) , \tag{2.15}$$

while the $m = n = 0$ term is just the original kernel (2.13), i.e.

$$W_,(;;x,y) \equiv W_{xy} . \tag{2.16}$$

For two supermatrix representations $\mathbf{v}(y)$ and $\mathbf{u}(x)$ we define in precise analogy, the supergauge invariant

$$\begin{aligned} \mathbf{u}\{W\}\mathbf{v} = \sum_{m,n=0}^{\infty} \int & d^D x\, d^D y\, d^D x_1 \cdots d^D x_n\, d^D y_1 \cdots d^D y_m \times \\ & \times W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(x_1,\ldots,x_n;y_1,\ldots,y_m;x,y) \times \\ & \times \mathrm{str}\Big[\mathbf{u}(x)\, \mathcal{A}_{\mu_1}(x_1)\cdots \mathcal{A}_{\mu_n}(x_n)\, \mathbf{v}(y)\mathcal{A}_{\nu_1}(y_1)\cdots \mathcal{A}_{\nu_m}(y_m)\Big] . \end{aligned} \tag{2.17}$$

Note that the superwine's vertices $W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}$ are the same as those in (2.14). Equation (2.17) corresponds precisely to the 'peppering' prescription in ref. [3]!

We restrict, as in ref. [1], to covariantizations which traverse back and forth along a coincident Wilson line, i.e. such that they may be represented as

$$\mathbf{u}\{W\}\mathbf{v} = \int\!\!\int d^D x\, d^D y \int \mathcal{D}\mathcal{C}_{xy}\, \mathrm{str}\, \mathbf{u}(x)\, \Phi[\mathcal{C}_{xy}]\, \mathbf{v}(y)\, \Phi^{-1}[\mathcal{C}_{xy}] . \tag{2.18}$$

Here the measure $\mathcal{D}\mathcal{C}_{xy}$ over curves $\mathcal{C}_{xy}$ from x to y, is normalised by

$$\int \mathcal{D}\mathcal{C}_{xy} 1 = W_{xy} , \tag{2.19}$$

and encodes our choice of covariantization. The (super) Wilson lines are defined by the path ordered exponential

$$\Phi[\mathcal{C}_{xy}] = P\exp -i\int_{\mathcal{C}_{xy}} dz^\mu \mathcal{A}_\mu(z)\,,$$
$$= 1 - i\int_0^1 d\tau\, \dot{z}^\mu \mathcal{A}_\mu(z) - \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1\, \dot{z}\cdot\mathcal{A}(\tau_1)\,\dot{z}\cdot\mathcal{A}(\tau_2) + \cdots\,, \qquad (2.20)$$

where we have parametrized $\mathcal{C}_{xy}$ by $z^\mu(\tau)$, $\tau \in [0,1]$, $z(0) = x$, $z(1) = y$. Since this definition is independent of the parametrization of the path, the same is true of the measure $\mathcal{D}\mathcal{C}_{xy}$, without loss of generality. We also choose the measure to be Lorentz covariant, satisfy the exchange symmetry below (2.14), and to be smooth in momentum space, i.e. to yield vertices that are Taylor expandable to all orders in momenta [1].

One example that satisfies all these criteria is to utilise the momentum representation to write [1]:

$$\mathbf{u}\{W\}\mathbf{v} = \mathrm{str}\int d^Dx\,\mathbf{u}(x)\,W\left(-\frac{\nabla^2}{\Lambda^2}\right)\cdot\mathbf{v}(x)\,. \qquad (2.21)$$

This is the covariantization that we will use in this paper to calculate the one-loop β function. (We will see later that it corresponds to coincident lines (2.18).) Another example is simply to use two straight super Wilson lines [1]:

$$\mathbf{u}\{W\}\mathbf{v} = \int\!\!\int d^Dx\, d^Dy\, W_{xy}\,\mathrm{str}\,\mathbf{u}(x)\Phi[l_{xy}]\mathbf{v}(y)\Phi^{-1}[l_{xy}]\,, \qquad (2.22)$$

l_{xy} being the straight line between x and y. We will compute concrete formulae for the wine vertices in both covariantizations.

Introducing the superfield strength $\mathcal{F}_{\mu\nu} = i[\nabla_\mu, \nabla_\nu]$, we write the 'seed' action [3] as

$$\hat{S} = \frac{1}{2}\mathcal{F}_{\mu\nu}\{c^{-1}\}\mathcal{F}_{\mu\nu} + \Lambda^2\,\mathcal{B}_\mu\{\tilde{c}^{-1}\}\mathcal{B}_\mu + \sigma\Lambda^4\,\mathbf{C}\{1\}\mathbf{C}\,. \qquad (2.23)$$

The first term, which we will refer to as $\hat{S}_A$, is simply the $\hat{S}$ of ref. [1] translated to superfields (corresponding again to peppering [3]), in particular $c(p^2/\Lambda^2) > 0$ is a smooth cutoff function satisfying $c(0) = 1$ (in fact without loss of generality) and $c(x) \to 0$ as $x \to \infty$.[5] In addition we have two further terms for $\mathbf{B}$ and $\mathbf{C}$, where

$$\mathcal{B}_\mu = \mathbf{B}_\mu + \nabla_\mu\cdot\mathbf{C}\,, \qquad \mathbf{C}\{1\}\mathbf{C} \equiv \mathrm{str}\int d^Dx\,\mathbf{C}^2(x)\,, \qquad (2.24)$$

and $\tilde{c}(p^2/\Lambda^2) > 0$ is another smooth cutoff profile whose properties [3] are recalled below. Eq. (2.23) is nothing but the seed action of ref. [3], translated into this language.

[5]As in refs. [1, 3], x used as a generic argument for these functions, should not be confused with the position x and the position space kernel c_{xy}, defined as in (2.13).

Note however that such terms are expected in spontaneously broken $\mathrm{SU}(N|N)$; thus $\hat{S}_A$ is just the higher-derivative regularised pure gauge part, in particular the wrong sign A_μ^2 action [3] is now seen to be a consequence of the superfield (in particular supertrace) structure; the second term, which we name $\hat{S}_B$, collects the remaining kinetic pieces after expanding around $\langle\mathcal{C}\rangle$ and imposing unitary gauge,[6] in particular providing the mass term for $\mathbf{B}$; the last term, $\hat{S}_C$, is the mass term for the 'Higgs'. (Here $\sigma > 0$ a free parameter. Note that the $\mathbf{C}$ field we discussed earlier appears here as $\Lambda\mathbf{C}$. This change of variables is required for the finiteness of the flow equation, as explained in section 9.) Recall that the point of $\mathbf{C}$ was to cancel the longitudinal divergences from $\mathbf{B}_\mu$ [3]. In the $\mathrm{SU}(N|N)$ language this cancellation is unsurprising because the longitudinal part of $\mathbf{B}_\mu$ is nothing but $\mathbf{D}$, the eaten fermionic partner of $\mathbf{C}$.

Actually for $\hat{S}_B$ to correspond exactly to spontaneously broken $\mathrm{SU}(N|N)$, the $\mathbf{B}$-$\mathbf{C}$ cross-terms should contain an insertion of $i\sigma_3$ arising from an explicit $\langle\mathcal{C}\rangle$. This in turn leads to some differences in the discrete symmetries here, as explained in section 5. Also, the self-interactions of the Higgs are missing: these are not needed for the regularisation (to one loop with only external gauge fields).

We require $\tilde{c}(x) \to 0$ as $x \to \infty$, to regularise $\mathbf{C}$ propagation, while for the $\tilde{c}^{-1}$ term not to disturb the high energy behaviour of the transverse part of $\mathbf{B}_\mu$ we clearly require

$$\frac{c}{(x\tilde{c})} \to 0 \qquad \text{as} \qquad x \to \infty\,. \tag{2.25}$$

(We require $\tilde{c}_0 \equiv \tilde{c}(0) > 0$ in order for it to act as a mass for $\mathbf{B}_\mu$ and positive kinetic term for $\mathbf{C}$ but we do not require $\tilde{c}_0 = 1$.) More precise requirements on the UV asymptotics of c and $\tilde{c}$ will be needed and are derived in section 9. We introduce three new kernels via

$$K(x) = \frac{d}{dx}\left(\frac{x\tilde{c}c}{x\tilde{c}+c}\right), \quad xL(x) = \frac{d}{dx}\left(\frac{x^2\tilde{c}^2}{x\tilde{c}+c}\right) \quad \text{and} \quad xM(x) = \frac{d}{dx}\left(\frac{x^2\tilde{c}}{x+\sigma\tilde{c}}\right) \tag{2.26}$$

($c \equiv c(x)$, $\tilde{c} \equiv \tilde{c}(x)$ here). Finally we can write our full exact RG equation:

$$\begin{aligned}\Lambda\frac{\partial}{\partial\Lambda}S[\mathbf{A},\mathbf{B},\mathbf{C}] = {} & \frac{1}{2\Lambda^2}\left(\frac{1}{N}\frac{\delta}{\delta\mathcal{A}_\mu} - \frac{\delta S}{\delta\mathcal{A}_\mu}\right)\{c'\}\frac{\delta\Sigma_g}{\delta\mathcal{A}_\mu} + \frac{1}{2\Lambda^4}\left(\frac{l}{N} - lS\right)\{L\}l\Sigma_g + \\ & + \frac{1}{2\Lambda^2}\left(\frac{1}{N}\frac{\delta}{\delta\mathbf{B}_\mu} - \frac{\delta S}{\delta\mathbf{B}_\mu}\right)\{K - c'\}_{\mathbf{A}}\frac{\delta\Sigma_g}{\delta\mathbf{B}_\mu} + \\ & + \frac{1}{2\Lambda^4}\left(\frac{1}{N}\frac{\delta}{\delta\mathbf{C}} - \frac{\delta S}{\delta\mathbf{C}}\right)\{M - L\}\frac{\delta\Sigma_g}{\delta\mathbf{C}},\end{aligned} \tag{2.27}$$

where

$$l = \frac{\delta}{\delta\mathbf{C}} + \nabla_\mu\cdot\frac{\delta}{\delta\mathcal{A}_\mu} \qquad \text{and} \qquad \Sigma_g = g^2S - 2\hat{S}\,. \tag{2.28}$$

[6]Apart from one significant difference described below

In here, the first term on the r.h.s. is just our pure gauge field flow equation [1] with A_μ replaced by $\mathcal{A}_\mu$ and trace by supertrace. Finiteness considerations motivated the precise form of the other terms (see section 9 and ref. [3].) By the wine $\{K-c'\}_{\mathbf{A}}$ we mean that in (2.17) only $\mathbf{A}$ is used, rather than the full peppered $\mathcal{A}$. In l we apply the $\delta/\delta\mathcal{A}_\mu$ first and then ∇_μ, i.e. we understand this $\delta/\delta\mathcal{A}_\mu$ as not differentiating the $\mathcal{A}_\mu$ in the ∇_μ of $\nabla_\mu\cdot\delta/\delta\mathcal{A}_\mu$. Note that as in ref. [1], by prime (as in c') we mean differentiation with respect to its argument (here p^2/Λ^2). The position space representation (2.13), the covariantization, and the resulting vertices (2.14), are all labelled by the underlying kernel (i.e. in this case replacing the letter W by c' throughout (2.13)–(2.17)). (2.27) yields precisely the Feynman rules chosen earlier for the β function calculation [3].

The coupling g is defined as in ref. [1] via the field strength for A^1_μ in the A^1 part of the action:

$$S|_{A^1}=\frac{1}{2g^2}\,\mathrm{tr}\int d^Dx\ \left(F^1_{\mu\nu}\right)^2+O(\partial^3) \tag{2.29}$$

(discarding the vacuum energy). Note that we can and do impose, as in ref.[1], the requirement of 'quasilocality', and thus in particular that S has a derivative expansion to all orders. (N.B. Other than of course its supersymmetry, this is the crucial fundamental property assumed of the wilsonian effective action in supersymmetric theories, which in turn justifies its holomorphy [4].)

Clearly, the flow equation (2.27) is manifestly $\mathrm{SU}(N)\times\mathrm{SU}(N)$ gauge invariant. It also leaves the partition function

$$\mathcal{Z}=\int\mathcal{D}[\mathbf{A},\mathbf{B},\mathbf{C}]\,\mathrm{e}^{-NS}$$

invariant. To see this, note that (2.27) implies that

$$\begin{aligned}\Lambda\frac{\partial}{\partial\Lambda}\mathrm{e}^{-NS}=&-\frac{1}{2\Lambda^2}\frac{\delta}{\delta\mathcal{A}_\mu}\{c'\}\left(\frac{\delta\Sigma_g}{\delta\mathcal{A}_\mu}\mathrm{e}^{-NS}\right)-\frac{l}{2\Lambda^4}\{L\}\left(\mathrm{e}^{-NS}l\Sigma_g\right)-\\&-\frac{1}{2\Lambda^2}\frac{\delta}{\delta\mathbf{B}_\mu}\{K-c'\}_{\mathbf{A}}\left(\frac{\delta\Sigma_g}{\delta\mathbf{B}_\mu}\mathrm{e}^{-NS}\right)-\frac{1}{2\Lambda^4}\frac{\delta}{\delta\mathbf{C}}\{M-L\}\left(\frac{\delta\Sigma_g}{\delta\mathbf{C}}\mathrm{e}^{-NS}\right),\end{aligned} \tag{2.30}$$

and hence is a total functional derivative. Actually, for this to be true we need the $\delta/\delta\mathcal{A}_\mu$ in the leftmost l to act on everything in the expression and thus also on the $\mathcal{A}_\mu$ in ∇_μ, in apparent contradiction with the definition given for (2.27). However, the difference between the two definitions gives $\frac{\delta}{\delta\mathcal{A}_\mu}\mathcal{A}_\mu$, which vanishes by the supertrace mechanism. (To see this set $\mathbf{Y}=1$ in (2.8).)

We can see indirectly that the exact RG equation (2.27) corresponds to integrating out by the arguments already given in ref. [1].

Similarly from [1], we still have that S may be expanded in traces and products of traces. We can write these as products of supertraces of the fields $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, and if necessary from (2.11) or (2.12), embedded σ_3's. Representing the supertraces

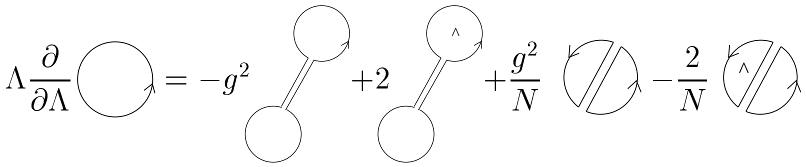


Figure 1: Diagrammatic representation of the flow equation. A circumflex in a circle indicates $\hat{S}$.

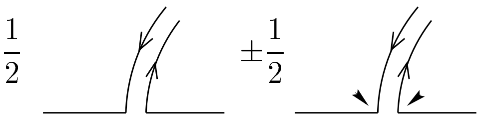


Figure 2: Insertion of σ_3s occurs in the same way, whether or not the attaching wine results in a tree or loop correction.

by closed loops we have the same diagrammatic notation for the full RG equations as before (figure 1), where here we have taken the (naïve [1]) large-N limit. Once again this implies that at most a single (super)trace survives in the effective action.

The expansion of the loops in powers of the fields yields Feynman diagrams as before [1]. Note that the closed loops may no longer necessarily be interpreted as integrals over pure gauge (super) Wilson loops because $\mathbf{C}$s, σ_3s and individual $\mathbf{B}$s may be inserted. However, the contributions consist of pure gauge (viz. $\mathcal{A}$) sections joining isolated $\mathbf{C}$s, σ_3s and $\mathbf{B}$s, and thus may be interpreted as integrals over pure gauge Wilson lines joining a countable number of such points.

If we represent diagrammatically the insertion of a σ_3 into (a join in) such a superloop, by a filled arrow pointing to the insertion point, the supersowing special cases (2.11), and the supersplitting special cases (2.12), can be seen to be caused by the same local process, as shown in figure 2

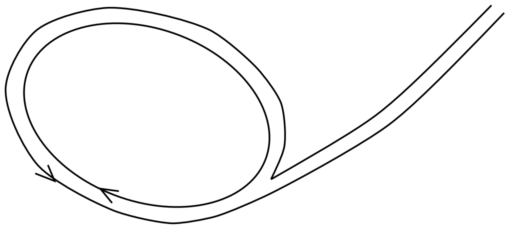


Figure 3: A wine biting its own tail.

Contributions to (2.27) containing terms of form of figure 3 are apparently required by (2.30) (where the wine attaches at the top to S or $\hat{S}$). However, in the large-N limit these contributions vanish. The reason is as follows. Firstly the $\{K - c'\}_{\mathbf{A}}$, or $\{M - L\}$, wines obviously cannot bite their own tails since they do not contain $\mathbf{B}$, or $\mathbf{C}$, respectively. Secondly the other end of the wine in figure 3 must attach to a vertex with at least two points, because all one-point vertices vanish by charge conjugation invariance (see section 6). Since only one superloop can contain fields in the large-N limit, the closed superloop in figure 3 must thus be field free. But for the two remaining possibilities $\{c'\}$ and $\{L\}$, this loop is formed by a 'full' supermatrix differential $\delta/\delta\mathcal{A}$ which thus yields $\mathrm{str}\, 1 = 0$ by (2.8).

3. Perturbative expansion

We recall the general structure of the perturbative expansion [1]. As in ref. [1] we keep D general at this stage. It will be helpful to access $D = 4$ via the limit $D \to 4$ [3]. It will also be helpful to write (2.27) as

$$\Lambda \frac{\partial}{\partial \Lambda} S = -a_0[S, g^2 S - 2\hat{S}] + a_1[g^2 S - 2\hat{S}]\,, \tag{3.1}$$

where we have expanded Σ_g and written the classical terms as the bilinear functional $-a_0$ and the quantum terms as the linear functional a_1. Since this has the same form as the pure gauge case, the isolation of perturbative contributions from these equations proceeds as before [1]. Thus we see from (3.1), that $S \sim 1/g^2$ at the classical level (consistent with (2.29)), and by iteration, using (3.1), that S has as expected the weak coupling expansion

$$S = \frac{1}{g^2} S_0 + S_1 + g^2 S_2 + \cdots\,. \tag{3.2}$$

Substituting this expansion in (3.1) and recalling that g will run at the quantum level, we see that the β function must also take the standard form

$$\beta := \Lambda \frac{\partial g}{\partial \Lambda} = \beta_1 g^3 + \beta_2 g^5 + \cdots\,. \tag{3.3}$$

By (2.29), g^2 has dimension $4 - D$. The β_i thus carry dimensions $(D-4)i$ and (as we will confirm) are not universal except when $D = 4$. From (3.2) and (3.3), we obtain the loopwise expansion of (3.1):

$$\Lambda \frac{\partial}{\partial \Lambda} S_0 = -a_0[S_0, S_0 - 2\hat{S}] \tag{3.4}$$

$$\Lambda \frac{\partial}{\partial \Lambda} S_1 = 2\beta_1 S_0 - 2a_0[S_0 - \hat{S}, S_1] + a_1[S_0 - 2\hat{S}] \tag{3.5}$$

$$\Lambda \frac{\partial}{\partial \Lambda} S_2 = 2\beta_2 S_0 - 2a_0[S_0 - \hat{S}, S_2] - a_0[S_1, S_1] + a_1[S_1]\,, \tag{3.6}$$

etc. [1]. In a similar way we obtain equations for the weak coupling expansion of integrated operators which in this notation are identical to those of ref. [1].

4. Feynman rules

This section sets out the nomenclature we will use for the Feynman rules and the resulting expressions as derived by expanding (2.27) in a power series in the fields. Along the way we explain, pedantically, the precise method for translating figure 1 to equations for individual vertices. As before [3, 1], this is diagrammatically represented by replacing the thick black lines of figure 1, which represent the exact

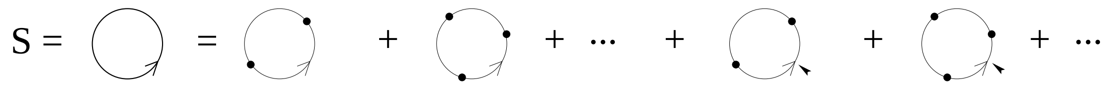


Figure 4: Expansion of the action into supertraces of fields, with and without a σ_3 insertion.

expressions without expansion in the fields, by thin lines, and a series of contributions with increasing numbers of points (blobs). The points represent individual fields and appear in all places on the composite Wilson loop with equal weight. Where necessary the points will now be labelled by the flavour of the field that attaches to it. Specializing to a single supertrace as appropriate for the large-N limit, the field expansion of the effective action, which is illustrated diagrammatically in figure 4 takes the form:

$$S = \sum \frac{1}{s_n} \int d^D x_1 \cdots d^D x_n \, S^{\sigma^j \mathbf{X}_1 \cdots \mathbf{X}_n}_{\;\;\;\; a_1 \cdots a_n}(x_1, \ldots, x_n) \, \mathrm{str}\, \sigma_3^j \mathbf{X}_1^{a_1}(x_1) \cdots \mathbf{X}_n^{a_n}(x_n) \,, \quad (4.1)$$

where the superfields $\mathbf{X}_i^{a_i}$ are $\mathbf{A}^{\mu_i}$, $\mathbf{B}^{\mu_i}$ or $\mathbf{C}$, the indices $a_i = \mu_i$, or null for $\mathbf{C}$, and $j = 0$ or 1. (We will omit the σ superscript for $j = 0$ and write it without exponent when $j = 1$.) Only one cyclic ordering of each list $\mathbf{X}_1, \ldots, \mathbf{X}_n$ appears in the sum. Furthermore, if the list $\mathbf{X}_1, \ldots, \mathbf{X}_n$ is invariant under some nontrivial cyclic permutations, then s_n is the order of the cyclic subgroup, otherwise $s_n = 1$. (See section 6.1.) In the process of computing S, there can arise any number of trapped σ_3s via figure 2, however in the field expansion these can always be reduced to at most one by (anti)commutation through ($\mathbf{B}$), $\mathbf{A}$ or $\mathbf{C}$, and if one remains it will be placed at the beginning of the supertrace (thus resulting in a marked superloop as illustrated).

Concentrating on a given vertex, but with j so far undetermined, and turning to the composite Wilson loops on the r.h.s. of figure 1, we assign the vertex's flavours ($\mathbf{A}$, $\mathbf{B}$ or $\mathbf{C}$) together with their associated momenta and if appropriate Lorentz indices, to the points, summing over all cyclic permutations. For each resulting configuration, the flavours of the component wine and action vertices can then be determined. The position of any embedded σ_3s in component action vertices then follow. Of course some cases already vanish at this stage due to the absence of the appropriate vertex, e.g. when a $\mathbf{C}$-point is placed on a wine. In other cases there may be more than one choice for the component vertices, in which case the corresponding Feynman diagrams are summed over. Attachments made via partial supermatrices are expanded as in figure 2. The full set of σ_3s can be combined inside the Wilson loops, (anti)commuting past external fields as necessary, and eliminated via $\sigma_3^2 = 1$ and/or $\mathrm{str}\, \sigma_3 = 2N$, leaving at most one σ_3, which is moved to its canonical position as determined by (4.1). (Actually, the expansion step figure 2, and all σ_3s can be ignored for classical vertices as explained in section 7.) Finally, applying momentum conservation and including loop momentum integrals if appropriate, we

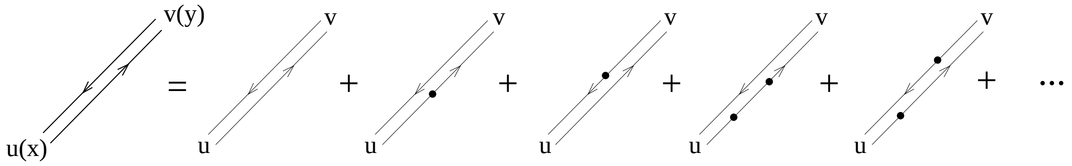


Figure 5: Expansion of the wine in gauge fields.

read off according to the Feynman rules set out below, the complete expressions for the component vertices, and multiply the whole by $1/2\Lambda^2$ [1].

The wine expansion (2.17) appears as figure 5. The wine component of the vertices is then given by (2.17) (with $\mathcal{A}$ replaced by $\mathbf{A}$ in the case of $\{K-c'\}_{\mathbf{A}}$) and thus in general, from (2.18)–(2.20), as [1]:

$$
\begin{aligned}
W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(p_1,\ldots,p_n;q_1,\ldots,q_m;r,s)\,(2\pi)^D\delta\left(\sum_{i=1}^{n}p_i+\sum_{j=1}^{m}q_j+r+s\right) = \\
= (-i)^{n+m}\int\!\!\int d^Du\,d^Dv\int\mathcal{D}\mathcal{C}_{uv}\int_u^v dx_n^{\mu_n}\,\times \\
\times\int_u^{x_n}dx_{n-1}^{\mu_{n-1}}\cdots\int_u^{x_2}dx_1^{\mu_1}\int_v^{u}dy_m^{\nu_m}\int_v^{y_m}dy_{m-1}^{\nu_{m-1}}\cdots\int_v^{y_2}dy_1^{\nu_1}\,\times \\
\times\exp -i\left(r\cdot u+s\cdot v+\sum_i p_i\cdot x_i+\sum_j q_j\cdot y_j\right),
\end{aligned}
\tag{4.2}
$$

where the x_i integration is along the curve $\mathcal{C}_{uv}$, and the y_j integration along the same curve but in the opposite direction cf. figure 6. (As in ref. [1], all momenta are taken to be pointing in to the vertex.) Explicit expressions up to $O(\mathcal{A}^2)$, for two covariantizations, are given in section 5.

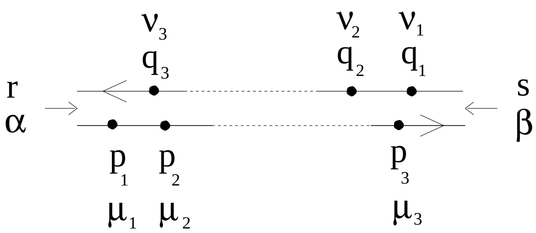


Figure 6: Feynman rule for the wine, with momentum and Lorentz labels.

From (2.27), we now have several extra pieces to take into account in the full Feynman rule for the wine vertices compared to ref. [1]: extra factors of $1/\Lambda^2$, contributions from l, the flavour and (if it exists) Lorentz index of the functional derivative the wine vertex actually attaches to. (Due to l, it is no longer the case that one just contracts these Lorentz indices.) A complete notation for the wine vertex Feynman rule can therefore be given as:

$$
V^{\mathbf{X}_1\cdots\mathbf{X}_n,\mathbf{Y}_1\cdots\mathbf{Y}_m,\mathbf{Z}_1\mathbf{Z}_2}_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m,a_1a_2}(p_1,\ldots,p_n;q_1,\ldots,q_m;r,s)\,, \tag{4.3}
$$

with momenta and indices as represented in figure 6, where the $\mathbf{X}_i^{\mu_i}(p_i)$ and $\mathbf{Y}_j^{\nu_j}(q_j)$ are $\mathbf{A}$s or $\mathbf{B}$s, and the functional derivatives are with respect to $\mathbf{Z}_1^{a_1}(r)$ and $\mathbf{Z}_2^{a_2}(s)$. The index a_1 (a_2) is the Lorentz index α (β) if $\mathbf{Z}_1$ ($\mathbf{Z}_2$) is $\mathbf{A}$ or $\mathbf{B}$, and null if it is $\mathbf{C}$.

This notation is convenient for listing the Feynman rules below. They may all be expressed immediately in terms of the expansion of L with its extra pieces. For this we write (with the momentum arguments $(p_1,\ldots,p_n;q_1,\ldots,q_m;r,s)$ here suppressed):

$$W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m,\alpha\beta} = \delta_{\alpha\beta}W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m} + L_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m,\alpha\beta}\,, \tag{4.4}$$

where shorthands (2.15) and (2.16) apply (see also (2.13) and above), and thus in particular at the zero-point level $W_{p,\alpha\beta} = \delta_{\alpha\beta}W_p + L_{p,\alpha\beta}$. The first few $\mathbf{C}$-$\mathbf{A}_\beta$ and $\mathbf{A}_\alpha$-$\mathbf{A}_\beta$ vertices extracted from $l\{L\}l$ are:

$$\begin{aligned}
L_{p,\beta} \equiv L_{,,\ \ \beta}(;\,;p,-p) &= \frac{ip_\beta L_p}{\Lambda^2} \\
L_{p,\alpha\beta} &= \frac{p_\alpha p_\beta L_p}{\Lambda^2} \\
L_{\mu,\beta}(p;q,r) &= \frac{-i\left\{r_\beta L_\mu(p;q,r) + \delta_{\mu\beta}L_q\right\}}{\Lambda^2} \\
L_{\mu,\alpha\beta}(p;q,r) &= \frac{\left\{r_\beta\delta_{\mu\alpha}L_r - q_\alpha\delta_{\mu\beta}L_q - q_\alpha r_\beta L_\mu(p;q,r)\right\}}{\Lambda^2} \\
L_{\mu\nu,\beta}(p,q;r,s) &= \frac{-i\left\{s_\beta L_{\mu\nu}(p,q;r,s) + \delta_{\beta\nu}L_\mu(p;r,s{+}q)\right\}}{\Lambda^2} \\
L_{\mu\nu,\alpha\beta}(p,q;r,s) &= \frac{1}{\Lambda^2}\Big\{\delta_{\mu\alpha}\delta_{\nu\beta}L_{p+r} + s_\beta\delta_{\mu\alpha}L_\nu(q;p{+}r,s) - \\
&\qquad - r_\alpha\delta_{\nu\beta}L_\mu(p;r,s{+}q) - r_\alpha s_\beta L_{\mu\nu}(p,q;r,s)\Big\}\,,
\end{aligned} \tag{4.5}$$

and via the coincident line identities (cf. section 5 and ref. [1]):

$$\begin{aligned}
L_{\mu,\nu,\beta}(p;q;r,s) &= -L_{\mu\nu,\beta}(p,q;r,s) - L_{\nu\mu,\beta}(q,p;r,s) \\
L_{\mu,\nu,\alpha\beta}(p;q;r,s) &= -L_{\mu\nu,\alpha\beta}(p,q;r,s) - L_{\nu\mu,\alpha\beta}(q,p;r,s)\,.
\end{aligned} \tag{4.6}$$

From (2.27), we thus obtain for the zero-point wine Feynman rules (4.3):

$$\begin{aligned}
V^{,,\mathbf{AA}}_{,,\,\mu\,\nu}(;\,;p,-p) &= c'_{p,\alpha\beta}\,, &\qquad V^{,,\mathbf{BB}}_{,,\,\alpha\,\beta}(;\,;p,-p) &= K_{p,\alpha\beta}\,, \\
V^{,,\mathbf{CC}}_{,,}(;\,;p,-p) &= \frac{M_p}{\Lambda^2}\,, &\qquad V^{,,\mathbf{CA}}_{,,\quad\beta}(;\,;p,-p) &= L_{p,\beta}\,,
\end{aligned} \tag{4.7}$$

and for the one-point wine Feynman rules, using (2.15):

$$\begin{aligned}
V^{\mathbf{A},\mathbf{AA}}_{\mu,\,\alpha\,\beta}(p;q,r) &= V^{\mathbf{B},\mathbf{AB}}_{\mu,\,\alpha\,\beta}('') = V^{\mathbf{B},\mathbf{BA}}_{\mu,\,\alpha\,\beta}('') = c'_{\mu,\alpha\beta}(p;q,r) \\
V^{\mathbf{A},\mathbf{BB}}_{\mu,\,\alpha\,\beta}(p;q,r) &= K_{\mu,\alpha\beta}(p;q,r) \\
V^{\mathbf{A},\mathbf{CC}}_{\mu,}(p;q,r) &= \frac{1}{\Lambda^2}M_\mu(p;q,r) \\
V^{\mathbf{A},\mathbf{CA}}_{\mu,\quad\beta}(p;q,r) &= V^{\mathbf{B},\mathbf{CB}}_{\mu,\quad\beta}('') = L_{\mu,\beta}(p;q,r) \\
V^{\mathbf{A},\mathbf{AC}}_{\mu,\,\alpha}(p;q,r) &= V^{\mathbf{B},\mathbf{BC}}_{\mu,\,\alpha}('') = -L_{\mu,\alpha}(p;r,q)\,.
\end{aligned} \tag{4.8}$$

Of course we do not list those that vanish due to the absence of an appropriate vertex. Similarly for the two-point wine Feynman rules:

$$
\begin{aligned}
V^{\mathbf{AA},\mathbf{AA}}_{\mu\nu,\alpha\beta}(p,q;r,s) &= V^{\mathbf{BA},\mathbf{BA}}_{\mu\nu,\alpha\beta}('') = V^{\mathbf{BB},\mathbf{AA}}_{\mu\nu,\alpha\beta}('') = V^{\mathbf{AB},\mathbf{AB}}_{\mu\nu,\alpha\beta}('') = c'_{\mu\nu,\alpha\beta}(p,q;r,s) \\
V^{\mathbf{AA},\mathbf{BB}}_{\mu\nu,\alpha\beta}(p,q;r,s) &= K_{\mu\nu,\alpha\beta}(p,q;r,s) \\
V^{\mathbf{A},\mathbf{A},\mathbf{AA}}_{\mu,\nu,\alpha\beta}(p;q;r,s) &= V^{\mathbf{B},\mathbf{A},\mathbf{AB}}_{\mu,\nu,\alpha\beta}('') = V^{\mathbf{A},\mathbf{B},\mathbf{AB}}_{\mu,\nu,\alpha\beta}('') = c'_{\mu,\nu,\alpha\beta}(p;q;r,s) \\
V^{\mathbf{AA},\mathbf{CA}}_{\mu\nu,\ \beta}(p,q;r,s) &= L_{\mu\nu,\beta}(p,q;r,s) \\
V^{\mathbf{AA},\mathbf{AC}}_{\mu\nu,\alpha}(p,q;r,s) &= L_{\nu\mu,\alpha}(q,p;s,r) \\
V^{\mathbf{A},\mathbf{A},\mathbf{CA}}_{\mu,\nu,\ \beta}(p;q;r,s) &= L_{\mu,\nu,\beta}(p;q;r,s) \\
V^{\mathbf{AA},\mathbf{CC}}_{\mu\nu,}(p,q;r,s) &= \frac{1}{\Lambda^2}M_{\mu\nu}(p,q;r,s)\,.
\end{aligned}
\tag{4.9}
$$

Here also we only list those that we will need later. The expansion of $\hat{S}_A$ in (2.23) is pure $\mathcal{A}$ and we reserve unlabelled $\hat{S}$ vertices for this:

$$
\hat{S}_A = \sum_{n=2}^{\infty}\frac{1}{n}\int d^Dx_1\cdots d^Dx_n\,\hat{S}_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n)\ \operatorname{str}\mathcal{A}_{\mu_1}(x_1)\cdots\mathcal{A}_{\mu_n}(x_n)\,. \tag{4.10}
$$

This way the same explicit expressions as in refs. [1, 3] apply:

$$
\begin{aligned}
\hat{S}_{\mu\nu}(p) \equiv \hat{S}_{\mu\nu}(p,-p) &= 2\frac{\Delta_{\mu\nu}(p)}{c_p} \\
\hat{S}_{\mu\nu\lambda}(p,q,r) &= \frac{2}{c_p}(p_\lambda\delta_{\mu\nu}-p_\nu\delta_{\lambda\mu}) + 2c_\nu^{-1}(q;p,r)(p_\lambda r_\mu - p\cdot r\delta_{\lambda\mu}) + \text{cycles} \\
\hat{S}_{\mu\nu\lambda\sigma}(p,q,r,s) &= \frac{1}{c_{p+q}}(\delta_{\sigma\mu}\delta_{\lambda\nu}-\delta_{\lambda\mu}\delta_{\nu\sigma}) + 2c_\nu^{-1}(q;p,r+s)(p_\sigma\delta_{\lambda\mu}-p_\lambda\delta_{\sigma\mu}) + \\
&\quad + 2c_\sigma^{-1}(s;p,r+q)(p_\nu\delta_{\mu\lambda}-p_\lambda\delta_{\mu\nu}) + 2c_{\nu\lambda}^{-1}(q,r;p,s)(p_\sigma s_\mu - p\cdot s\delta_{\sigma\mu}) + \\
&\quad + c_{\nu,\sigma}^{-1}(q;s;p,r)(p_\lambda r_\mu - p\cdot r\delta_{\lambda\mu}) + \text{cycles}
\end{aligned}
\tag{4.11}
$$

etc., where in the two-point vertex we set $p_1 = -p_2 = p$, and introduce the transverse combination $\Delta_{\mu\nu}(p) := p^2\delta_{\mu\nu} - p_\mu p_\nu$, in the three-point vertex we add the two cyclic permutations of $(p_\mu, q_\nu, r_\lambda)$, and in the four-point vertex the three cyclic permutations of $(p_\mu, q_\nu, r_\lambda, s_\sigma)$.

The full 'seed' vertices that we will need, are then given as follows, cf. figure 7. Two-point vertices (there are no one-point vertices):

$$
\begin{aligned}
\hat{S}^{\mathbf{AA}}_{\mu\nu}(p) &= \hat{S}_{\mu\nu}(p) \\
\hat{S}^{\mathbf{BB}}_{\mu\nu}(p) &= \hat{S}_{\mu\nu}(p) + \frac{2\Lambda^2\delta_{\mu\nu}}{\tilde{c}_p} \\
\hat{S}^{\mathbf{CC}}(p) &= \frac{2\Lambda^2 p^2}{\tilde{c}_p} + 2\sigma\Lambda^4\,,
\end{aligned}
\tag{4.12}
$$

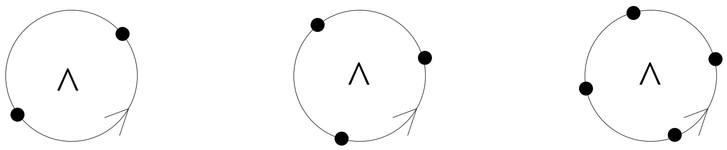


Figure 7: Feynman rules for the seed vertices.

three-point vertices:

$$
\begin{aligned}
\hat{S}^{\mathbf{AAA}}_{\mu\,\nu\,\lambda}(p,q,r) &= \hat{S}_{\mu\nu\lambda}(p,q,r)\\
\hat{S}^{\mathbf{CCA}}_{\lambda}(p,q,r) &= 2\Lambda^2\left\{p_\lambda\,\tilde{c}_p^{-1} - q_\lambda\,\tilde{c}_q^{-1} - p.q\,\tilde{c}_\lambda^{-1}(r;q,p)\right\}\\
\hat{S}^{\mathbf{BBA}}_{\mu\,\nu\,\lambda}(p,q,r) &= \hat{S}_{\mu\nu\lambda}(p,q,r) + 2\Lambda^2\delta_{\mu\nu}\,\tilde{c}_\lambda^{-1}(r;q,p)\\
\hat{S}^{\mathbf{BBC}}_{\mu\,\nu}(p,q,r) &= 2i\Lambda^2\left\{r_\nu\,\tilde{c}_\mu^{-1}(p;r,q) + r_\mu\,\tilde{c}_\nu^{-1}(q;p,r)\right\},
\end{aligned}
\tag{4.13}
$$

and four-point vertices:

$$
\begin{aligned}
\hat{S}^{\mathbf{AAAA}}_{\mu\,\nu\,\lambda\,\sigma}(p,q,r,s) &= \hat{S}_{\mu\nu\lambda\sigma}(p,q,r,s)\\
\hat{S}^{\mathbf{BBAA}}_{\mu\,\nu\,\lambda\,\sigma}(p,q,r,s) &= \hat{S}_{\mu\nu\lambda\sigma}(p,q,r,s) + 2\Lambda^2\delta_{\mu\nu}\,\tilde{c}_{\lambda\sigma}^{-1}(r,s;q,p)\\
\hat{S}^{\mathbf{CCAA}}_{\lambda\,\sigma}(\text{''}) &= 2\Lambda^2\Big\{\delta_{\lambda\sigma}\tilde{c}_{p+s}^{-1} + p_\lambda\tilde{c}_\sigma^{-1}(s;q+r,p) -\\
&\qquad - q_\sigma\tilde{c}_\lambda^{-1}(r;q,p+s) - p\cdot q\tilde{c}_\lambda^{-1}(r,s;q,p)\Big\}.
\end{aligned}
\tag{4.14}
$$

5. Explicit expressions for wine vertices

The wine vertices are defined implicitly by the choice of covariantization, and (2.14) or equivalently (2.17). In this section we derive formulae for the covariantizations (2.21) and (2.22), although in this paper we will only use (2.21) for concrete calculations.

5.1 Straight Wilson lines

In the straight line case (2.22), the integration over curves in (4.2), is trivial, being replaced by its normalisation (2.19), whilst $\mathcal{C}_{uv}$ is just the straight line which we may parametrize as $x_i = u + (v-u)t_i$ and $y_i = v + (u-v)t'_i$. Translation invariance implies that the integrand depends on u and v only in the combination $v - u$. We replace $v - u$ with derivatives with respect to the its conjugate momentum

$$
p = s + \sum_{i=1}^{n} p_i t_i + \sum_{j=1}^{m} q_j(1-t'_j)\,.
$$

(This expression drops out by substitution in the exponential of (4.2) and is unique up to the use of overall momentum conservation.) Integrating over the overall position yields the overall momentum conserving δ function. Cancelling this from both sides of (4.2) we arrive at

$$
\begin{aligned}
W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(p_1,\ldots,p_n;q_1,\ldots,q_m;r,s) = (-1)^m \int_0^1 dt_n \int_0^{t_n} dt_{n-1}\cdots\int_0^{t_2} dt_1 \times \\
\int_0^1 dt'_m \int_0^{t'_m} dt'_{m-1}\cdots\int_0^{t'_2} dt'_1 \times \\
\times \frac{\partial}{\partial p^{\mu_1}}\cdots\frac{\partial}{\partial p^{\mu_n}}\frac{\partial}{\partial p^{\nu_1}}\cdots\frac{\partial}{\partial p^{\nu_m}} W_p\,. \qquad (5.1)
\end{aligned}
$$

As an example, the one-point vertex is given by

$$
W^\mu(p;r,s) = \frac{2}{\Lambda^2}\int_0^1 dt\, W'_{tp+s}\,(tp+s)^\mu\,. \qquad (5.2)
$$

We note that these straight line vertices are completely symmetric on all Lorentz indices.

5.2 Covariantization via (2.21)

We first explain why this covariantization is of the coincident line form (2.18), equivalently (4.2). In expanding (2.21), or preferably (since the supermatrices are not necessary here)

$$
u\{W\}v = \mathrm{tr}\int d^Dx\, u(x) W\left(-\frac{D^2}{\Lambda^2}\right)\cdot v(x)\,, \qquad (5.3)
$$

in powers of the gauge field, we only need write

$$
\begin{aligned}
u\{W\}v = \sum_{m,n=0}^{\infty}\int d^Dx\, d^Dy\, d^Dx_1\cdots d^Dx_n\, W_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n;x,y)\times \\
\times\,\mathrm{tr}[u(x)\,A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\cdot v(y)]\,, \qquad (5.4)
\end{aligned}
$$

where these vertices are the coefficients of the ordered product of gauge fields $A_{\mu_1}(x_1)$ $\ldots A_{\mu_n}(x_n)$ as before, but moreover, each of these gauge fields is understood to act by commutation on the expression to its right. Expanding the commutators and relabelling the gauge fields that appear on the right hand side of $v(y)$ in terms of $A_{\nu_i}(y_i)$, as in (2.14), the other vertices are then given in terms of them by

$$
W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(p_1,\ldots,p_n;q_1,\ldots,q_m;r,s) = (-)^m \sum_{\text{interleaves}} W_{\lambda_1\cdots\lambda_{m+n}}(k_1,\ldots,k_{m+n};r,s)\,, \qquad (5.5)
$$

where we have (trivially) transferred to momentum space, and the sum runs over all interleaves of the sequences $p_1^{\mu_1},\ldots,p_n^{\mu_n}$ and $q_m^{\nu_m},\ldots,q_1^{\nu_1}$ i.e. combined sequences $k_1^{\lambda_1},\ldots,k_{m+n}^{\lambda_{m+n}}$ in which the p^μs remain ordered with respect to each other, and similarly the q^νs remain in reverse order. These are in fact the "coincident line" identities of ref. [1]. Now we employ a single-Wilson-line representation for the Ws on the r.h.s. of (5.5), i.e. (4.2) with $m = 0$. (This follows from a path integral representation of the kernel in (5.3), expressing the As in the adjoint representation.) By isolating the integrals over the coordinates conjugate to the q_js in the right hand terms of (5.5), we readily see that they collect together, and on reversing their direction, form the y_j integrals of the coincident line representation (4.2).

Now we compute the contribution to $W_{\mu_1\cdots\mu_n}(p_1,\dots,p_n;r,s)$ from the $(-D^2)^m$ term in the Taylor expansion of $W(-D^2/\Lambda^2)$. (This expansion must exist by quasilocality [1].) In this paper we only need explicit expressions to order A^2, but we derive a formula for the general vertex below, in order to analyse its large momentum behaviour. Expanding,

$$
\begin{aligned}
(-D^2)^m = (-\partial^2)^m + \sum_{\alpha=0}^{m-1}(-\partial^2)^\alpha(i\partial\cdot A + iA\cdot\partial + A^2)(-\partial^2)^{m-1-\alpha} - \\
- \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+\beta+\gamma=m-2}} (-\partial^2)^\alpha(\partial\cdot A + A\cdot\partial)(-\partial^2)^\beta(\partial\cdot A + A\cdot\partial)(-\partial^2)^\gamma + O(A^3)\,.
\end{aligned}
\tag{5.6}
$$

We already have noted in (2.16) that the zeroth order in A gives

$$
W(;r,s) = W_s\,, \tag{5.7}
$$

and this is trivially confirmed by (5.6). Transforming the $O(A)$ terms to momentum space, we see that (5.6) supplies a contribution to $W_\mu(p;r,s)$ of the form

$$
(p+2s)_\mu \sum_{\alpha=0}^{m-1} r^{2\alpha}s^{2m-2-2\alpha} = (r-s)_\mu \frac{r^{2m}-s^{2m}}{p\cdot(r-s)} \tag{5.8}
$$

(where we have used momentum conservation) and thus resumming the Taylor expansion,

$$
W_\mu(p;r,s) = (r-s)_\mu\,\frac{W_r - W_s}{p.(r-s)}\,. \tag{5.9}
$$

Similarly at $O(A^2)$, (5.6) supplies a contribution to $W_{\mu\nu}(p,q;r,s)$ of form

$$
\delta_{\mu\nu}\sum_{\alpha=0}^{m-1} r^{2\alpha}s^{2m-2-2\alpha} + (p+2s+2q)_\mu(q+2s)_\nu \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+\beta+\gamma=m-2}} r^{2\alpha}(s+q)^{2\beta}s^{2\gamma}\,. \tag{5.10}
$$

The latter sum may readily be evaluated e.g. by noting that it is equal to

$$
\frac{1}{2\pi i}\oint \frac{dz}{z^{m-1}}\,\frac{1}{(1-r^2z)(1-[s+q]^2z)(1-s^2z)}
$$

for a contour of infinitessimal radius encircling the origin (which we close onto the other poles), and thus after resumming the expansion of W,

$$
\begin{aligned}
W_{\mu\nu}(p,q;r,s) = \delta_{\mu\nu}\frac{W_s - W_r}{s^2 - r^2} - \\
-(p+2r)_\mu(q+2s)_\nu\Bigg\{\frac{W_{s+q}}{q\cdot(q+2s)p\cdot(p+2r)} + \\
+\frac{1}{s^2-r^2}\left[\frac{W_r}{p\cdot(p+2r)} - \frac{W_s}{q\cdot(q+2s)}\right]\Bigg\}.
\end{aligned}
\tag{5.11}
$$

The case where the gauge fields appear on either side of the wine is then given in terms of this by (5.5):

$$
W_{\mu,\nu}(p;q;r,s) = -W_{\mu\nu}(p,q;r,s) - W_{\nu\mu}(q,p;r,s)\,. \tag{5.12}
$$

5.3 Special momenta and covariantization (2.21)

The formulae (5.9) and (5.11) are ambiguous at certain special momenta. One way to determine the correct value is simply to return to first principles and resum (5.8) and (5.10) at the special values. However it is comforting to find that these results also appear uniquely by recalling that these formulae are only valid when the total momentum flowing into the vertex vanishes (i.e. is conserved), and taking the limit as the special configuration is approached. Thus (5.9) needs care at the point $p = 0$, i.e. $r = -s$, however by momentum conservation we can replace $p\cdot(r-s)$ by $s^2 - r^2$ after which the limit $r \to -s$ is trivial in (5.9), giving

$$
W_\mu(0;-s,s) = \frac{2s_\mu}{\Lambda^2}W'_s\,. \tag{5.13}
$$

(This also follows from resumming (5.8) with $p = 0$ and $r = -s$, or from the gauge transformation relations cf. (6.2) or ref. [1]:

$$
p^\mu W_\mu(p;-s-p,s) = W_{s+p} - W_s
$$

by expanding to first order in p after which p^μ may be removed uniquely from both sides. The longitudinal parts of higher powers in p are also determined uniquely by gauge invariance. See ref. [1, appendix A].) Note that the case $r = s$ ($p = -2s$) in (5.9) trivially gives zero (by a limit or from (5.8)).

Eq. (5.11) needs care at the point $p = -q$ corresponding to $r = -s$:

$$
\begin{aligned}
W_{\mu\nu}(p,-p;r,-r) &= \lim_{\epsilon\to 0} W_{\mu\nu}(p,-p-\epsilon;r+\epsilon,-r) \\
&= \frac{\delta_{\mu\nu}}{\Lambda^2}W'_r + (p+2r)_\mu(p+2r)_\nu\left\{\frac{W_{r+p}-W_r}{[p\cdot(p+2r)]^2} - \frac{W'_r}{\Lambda^2 p\cdot(p+2r)}\right\}.
\end{aligned}
\tag{5.14}
$$

There are other special momentum configurations that need careful definition in (5.11), but we will need only (5.14) here.

5.4 The general-point vertex in covariantization (2.21)

The order A^n term in $(-D^2)^m$ supplies a contribution to $W_{\mu_1\cdots\mu_n}(p_1,\ldots,p_n;r,s)$ which is a sum over a insertions of A^2 and b insertions of $\partial\cdot A + A\cdot\partial$ and over all permutations of these factors, cf. (5.6). In momentum space this reads

$$\sum_{\substack{a,b\\ 2a+b=n}} \sum_{\text{perms}} T_1^{i_1}\cdots T_{a+b}^{i_{a+b}} \sum_{\substack{\alpha_0,\ldots,\alpha_{a+b}\\ \sum\alpha_j=m-a-b}} \prod_{k=0}^{a+b} P_{I_k}^{2\alpha_k}\,, \qquad (5.15)$$

where the $i_k = 1$ or 2, according to whether a $\partial\cdot A + A\cdot\partial$ or A^2 term is taken from D^2 respectively. They yield respectively the tensors

$$T_k^1 = p_{I_k}^{\mu_{I_k}} + 2P_{I_k}^{\mu_{I_k}} \qquad \text{and} \qquad T_k^2 = \delta^{\mu_{I_k-1}\mu_{I_k}}\,.$$

$I_k = \sum_{j=1}^k i_j$ keeps track of the number of gauge fields accounted for, with $I_0 := 0$ and $I_{a+b} = n$, and the total momentum flow (directed from s) into the k^{th} insertion is given by

$$P_{I_k} = s + \sum_{j=I_k+1}^{n} p_j\,.$$

These definitions are illustrated in figure 8. Note that $P_{I_0} = P_0 = -r$ and $P_{I_{a+b}} = P_n = s$.

As in (5.10), the r.h. sum in (5.15) may be evaluated readily by writing first

$$\sum_{\substack{a,b\\ 2a+b=n}} \sum_{\text{perms}} T_1^{i_1}\cdots T_{a+b}^{i_{a+b}} \frac{1}{2\pi i}\oint \frac{dz}{z^{m+1-a-b}} \prod_{k=0}^{a+b} \frac{1}{1-zP_{I_k}^2}$$

for a contour of infinitessimal radius encircling the origin. Closing on to the other poles and resumming the Taylor expansion of W, we thus obtain

$$W_{\mu_1\cdots\mu_n}(p_1,\ldots,p_n;r,s) = \sum_{\substack{a,b\\ 2a+b=n}} \sum_{\text{perms}} T_1^{i_1}\cdots T_{a+b}^{i_{a+b}} \sum_{j=0}^{a+b} W_{P_{I_j}} \prod_{\substack{k=0\\ k\neq j}}^{a+b} \frac{1}{P_{I_j}^2 - P_{I_k}^2}\,. \qquad (5.16)$$

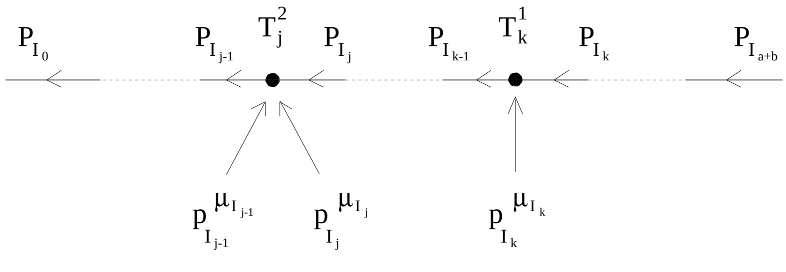


Figure 8: Tensor structure and momentum flow in the general vertex.

5.5 Large momentum behaviour of covariantization (2.21)

We determine the behaviour of such a vertex when a given momentum flow through the vertex becomes large. This is needed to study the finiteness properties of the exact RG and is used for the proof in section 9. We have already derived in ref. [1, appendix A], lower bounds on the divergences, by gauge invariance considerations. Simply by power counting, we can now obtain from (5.16), upper bounds on these divergences. In some cases, where no cancellations can occur or where the lower and upper bound agree, we can thus be confident that the large momentum behaviour is then precisely known. Furthermore, we can readily furnish simple expressions for the coefficients of the leading large momentum behaviour.

It is helpful in this section to adopt the following convention. By $\approx$ we will mean equal up to corrections that decay relative to the stated term as the large momentum $k \to \infty$; by $\sim$ we mean also that the stated term is determined only up to a coefficient of $O(k^0)$.

In fact this level of precision is much more than we need here, since the proof in section 9 only relies on the following properties: when the large momentum flow (the loop momentum k) is from end to end ($s \approx k$, $r \approx -k$, $k \to \infty$) the behaviour of the n point vertex is no worse than that of the zero point vertex W_k, and when the large momentum instead enters via the side (one $p_j \approx \pm k$) the 'covariantized differentiated propagators' $\{c'\}$, $\{K\}$, $\{L\}$, $\{M\}$ do not diverge.

Consider first the case where the large momentum flow is from end to end ($s \approx k$, $r \approx -k$, $k \to \infty$, all the p_j finite). In this case all the $P^2_{I_j} \approx k^2$, so that $P^2_{I_j} - P^2_{I_k} \sim k$. Including all the $T^1 \sim k$ ($T^2 \sim k^0$ of course), one may imagine from (5.16) that we simply have $W_{\mu_1\cdots\mu_n}(p_1, \ldots, p_n; r, s) \sim W_k$, where we have used the fact that the leading behaviour apparently arises from having $a = 0$. Whilst good enough for our purposes the ultraviolet behaviour is clearly better than this, since from (5.15), each term in the Taylor series goes as k^{2m-n} (independent of a and b) and thus as a whole:

$$W_{\mu_1\cdots\mu_n}(p_1, \ldots, p_n; r, s) \approx \frac{W_k}{k^n} \sum_{\substack{a,b \\ 2a+b=n}} 2^b \sum_{\text{perms}} \tilde{T}_1^{i_1} \cdots \tilde{T}_{a+b}^{i_{a+b}} \qquad (s \approx -r \approx k)\,, \quad (5.17)$$

where the tensor sum is $O(k^0)$, with $\tilde{T}^2 = T^2$, $\tilde{T}^1 = k^{\mu_{I_k}}/k$. Furthermore we note that this expression agrees with the lower bound from ref. [1], so we are confident that (5.17) is correct. The problem with the analysis of (5.16) is that when many terms are involved as here, there can be many cancellations in (5.16) of the leading behaviours as determined by power counting. But of course such power counting always yields an upper bound on the large momentum behaviour.

From (5.5) we readily find the large momentum behaviour for the $n+m$ point vertex in the case the large momentum $s \approx -r \approx k$ flows from end to end, in particular we see that it is $\sim W_k/k^{m+n}$ in agreement with the lower bound from [1].

For the more complex cases, the results depend on whether the kernel W_k decays or grows for large momentum. In the growing cases, corresponding in this paper to $W = c^{-1}$ or $\tilde{c}^{-1}$, we can again obtain a too pessimistic upper bound from power counting (5.16). The exact result, agreeing with the lower bound from [1], again follows most straightforwardly from (5.15). Thus, from the maximum power of k in (5.15), we see in particular that an $n+m$ vertex with large momentum k entering and leaving from any two points is again $\sim W_k/k^{m+n}$.

When the kernel W_k decays for large k (corresponding in this paper to c'_k, K_k, L_k and M_k) the leading large-k behaviour depends on the precise configuration. Thus we will show that

$$
\begin{aligned}
&W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(p_1,\ldots,p_n;q_1,\ldots,q_m;r,s) \approx \\
&\quad\approx -\frac{k^{\mu_n}}{k^2} W_{\mu_1\cdots\mu_{n-1},\nu_1\cdots\nu_m}(p_1,\ldots,p_{n-1};q_1,\ldots,q_m;r,s+p_n) \quad (s\approx -p_n\approx k) \qquad (5.18)\\
&\quad\approx \left(2\frac{k^{\mu_{n-1}}k^{\mu_n}}{k^4} - \frac{\delta^{\mu_{n-1}\mu_n}}{k^2}\right) W_{\mu_1\cdots\mu_{n-2},\nu_1\cdots\nu_m}(p_1,\ldots,p_{n-2};q_1,\ldots,q_m;r,s+p_{n-1}+p_n)\\
&\qquad\qquad (s\approx -p_{n-1}\approx k) \qquad (5.19)\\
&\quad\sim \frac{1}{k^{n-j+1}} W_{\mu_1\cdots\mu_{j-1},\nu_1\cdots\nu_m}\left(p_1,\ldots,p_{j-1};q_1,\ldots,q_m;r,s+\sum_{i=j}^{n}p_i\right) \quad (s\approx -p_j\approx k) \qquad (5.20)
\end{aligned}
$$

Closely similar identities then follow easily for k leaving via q_1, q_2 or any q_i, from the charge conjugation invariance identity

$$
\begin{aligned}
&W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(p_1,\ldots,p_n;q_1,\ldots,q_m;r,s) = \\
&\qquad = (-)^{n+m} W_{\nu_m\cdots\nu_1,\mu_n\cdots\mu_1}(q_m,\ldots,q_1;p_n,\ldots,p_1;r,s)\,, \qquad (5.21)
\end{aligned}
$$

equivalently reversing the direction of the Wilson lines in (4.2), [1].

For the cases where the large momentum both enters and leaves through the side, say $p_{j+1}\approx -p_j\approx k$, we will show that

$$
W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(p_1,\ldots,p_n;q_1,\ldots,q_m;r,s) \approx W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}\left[\delta^{\mu_j\mu_{j+1}} \mapsto \frac{\Delta^{\mu_j\mu_{j+1}}(k)}{k^2}\right], \qquad (5.22)
$$

where here the notation means to keep only those terms in $W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}$ which contain $\delta^{\mu_j\mu_{j+1}}$ and for these make the replacement indicated (no other changes required). In particular this allows us to read off from (5.11),

$$
W_{\mu\nu}(k,-k-r-s;r,s) \approx \frac{\Delta_{\mu\nu}(k)}{k^2}\frac{W_s - W_r}{s^2 - r^2} \qquad (5.23)
$$

and from (5.14),

$$
W_{\mu\nu}(k,-k;p,-p) \approx \frac{\Delta_{\mu\nu}(k)}{k^2\Lambda^2} W'_p\,.
$$

For general separation, on one side:

$$W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(p_1,\dots,p_n;q_1,\dots,q_m;r,s)\sim\frac{1}{k^{v-1}}\qquad(p_{j+v}\approx -p_j\approx k). \tag{5.24}$$

Clearly these conclusions hold also for the q side (e.g. from (5.21), or exchange or Lorentz symmetry, see section 6 or [1]). If k enters and leaves by different sides, say $p_i\approx -q_j\approx k$, then by (5.5), (5.24) and (5.22), we see that the vertex behaves as $\sim\Delta^{\mu_i\nu_j}(k)/k^2$, irrespective of the values of i and j (reflecting the fact that in (4.2) these points can always get close to each other).

These properties are arrived at, as follows. Setting $s\approx k$, $p_n\approx -k$ in (5.16), all other momenta fixed and finite, the leading term arises when $i_{a+b}=1$, through $T^1_{a+b}\approx k^{\mu_n}$. Since we can neglect the $j=a+b$ term, and factor out T^1_{a+b} and the denominator $P^2_{I_j}-P^2_{a+b}\approx -k^2$ in the $j\neq a+b$ terms, we are left precisely with the expression for the $n-1$ point vertex, giving (5.18) for $m=0$. Now if k leaves through $p_{n-1}\approx -k$, then the leading terms are furnished by $T^1_{a+b}T^1_{a+b-1}$ and T^2_{a+b}. Factoring out the now two divergent denominators gives the $m=0$ cases of (5.19). Proceeding similarly with the k leaving point further down the line, we readily find (5.20) for $m=0$. The large momentum behaviour of these three cases is confirmed by the fact that they saturate the lower bounds derived in ref. [1] (and in addition the first two clearly involve no cancellations). Now using (5.5), we readily see that they hold also for $m>0$ (the terms on the r.h.s. of (5.5) only contributing at leading order when no q momenta get between the divergent pair, p_j and s).

The cases where both large momenta are on the side, follow similarly. Thus with $p_{j+1}\approx -p_j\approx k$ in (5.16), we see that the leading terms come from $T^2=\delta^{\mu_j\mu_{j+1}}$ in which case no P_{I_k} diverges, and $(T^1)^2\approx k^{\mu_j}k^{\mu_{j+1}}$ with $-1/k^2$ from factoring out a divergent denominator. Since there is no opportunity for cancellation we can be confident that (5.22) is then correct, despite the fact that the gauge invariance analysis gives a lower bound at $\sim 1/k$ [1]. Indeed this disagreement between upper and lower bounds is allowed precisely because the leading behaviour (5.22) is transverse in k and thus 'escapes' the Ward identities. (5.24) follows from a similar analysis and agrees with gauge invariance analysis expectations after taking into account the more divergent initial case (5.22). Finally the $m>0$ cases are established as before, by use of (5.5).

We note that it is quite straightforward to go further and establish the order of the next-to-leading terms that we have been neglecting (typically down by $1/k$), and even their precise form.

5.6 Large momentum behaviour of straight line vertices

We finish this section with a brief remark about vertices (5.1). Broadly speaking, the straight line vertices have a comparable behaviour. However, let W_k decay for large k, and consider as an example (5.2), where the large momentum leaves through the side:

$$W^\mu(-k-r;r,k) = \frac{2}{\Lambda^2}\int_0^1 dt\, W'_{tk+(t-1)r}\,[tk+(t-1)r]^\mu\,.$$

We cannot just take the leading term from the t integrand because this leads to an integral that does not converge at $t=0$. Substituting $t=\Lambda\sqrt{x}/k$, we obtain the leading behaviour:

$$W^\mu(-k-r;r,k) = \frac{1}{k}\int_0^\infty \frac{dx}{\sqrt{x}}\, W'(\zeta^2)\,\zeta^\mu\,,$$

where $\zeta = \hat{k}\sqrt{x} - r/\Lambda$ (so $\zeta^2 = x - 2r\cdot\hat{k}/\Lambda + r^2/\Lambda^2$), with $\hat{k}$ being the unit vector in direction k, and corrections to the above being $O(1/k^2)$. We see that as in (5.18), $W_\mu \sim 1/k$, however unlike (5.18) and the other cases above, it cannot be expressed as an inverse power of k^2 with a coefficient which is analytic in its momenta (here k and r). Although mostly a matter of taste, it is the more regular large momentum behaviour of the vertices following from (2.21), that led us to use this covariantization for the concrete calculations reported in this paper.

6. Symmetries

As well as cyclic and exchange symmetry, and the symmetries of gauge invariance and charge conjugation, that are inherited and preserved from the formulation in ref. [1] (and may be interpreted geometrically in terms of Wilson loops [1, 3]), some new symmetries appear: fermion number, which is the U(1) remainder of the original global U($N|N$) symmetry, and an interesting Z_2 symmetry that exchanges the two SU(N) subgroups whilst effectively changing the sign of g^2. This latter symmetry is thus a "theory space" symmetry, a symmetry of the flow equation (2.27) but not of the action S.

These symmetries provide the key to understanding the formulation at a deeper level. We comment on them below, providing definitions where necessary. We also comment on reality, and include for later some comments on Poincaré invariance and dimensional assignments.

6.1 Cyclicity

Some action vertices inherit symmetries from cyclicity of the supertrace (for supermatrices). We have already mentioned this in section 4 where these vertices were defined divided by the order of the symmetry group. Thus the vertices in (4.10) are fully cyclically symmetric:

$$\hat{S}_{\mu_1\cdots\mu_n}(p_1,\dots,p_n) = \hat{S}_{\mu_2\cdots\mu_n\mu_1}(p_2,\dots,p_n,p_1)\,.$$

Similarly, the vertices (4.12) appear in $\hat{S}$ with a factor 1/2, and the **AA** and **BB** vertices are consequently symmetric under $\mu \leftrightarrow \nu$, while only the pure **A** vertices in the selections presented in (4.13) and (4.14), have any cyclic symmetry.

As mentioned below (4.1), and explained later, odd-loop contributions carry an insertion of σ_3. Since $\mathbf{A}$ and $\mathbf{C}$ commute with σ_3 but $\mathbf{B}$ anticommutes with σ_3, odd-loop vertices with $\mathbf{B}$ are antisymmetric under cyclic permutations that result in the original order of flavours but cycle an odd number of $\mathbf{B}$s. As an example, we see that the odd-loop contributions to $S^{\mathbf{BB}}_{\mu\,\nu}(p)$ vanish, because these must be antisymmetric under $\mu \leftrightarrow \nu$, but no such tensor can be constructed.

6.2 Exchange symmetry

From the comment below (2.14), we have

$$W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(p_1,\ldots,p_n;q_1,\ldots,q_m;r,s) = W_{\nu_1\cdots\nu_m,\mu_1\cdots\mu_n}(q_1,\ldots,q_m;p_1,\ldots,p_n;s,r)\,,$$

and similarly for the full vertices,

$$\begin{aligned} V^{\mathbf{X}_1\cdots\mathbf{X}_n,\mathbf{Y}_1\cdots\mathbf{Y}_m,\mathbf{Z}_1\mathbf{Z}_2}_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m,a_1a_2}&(p_1,\ldots,p_n;q_1,\ldots,q_m;r,s) = \\ &= V^{\mathbf{Y}_1\cdots\mathbf{Y}_m,\mathbf{X}_1\cdots\mathbf{X}_n,\mathbf{Z}_2\mathbf{Z}_1}_{\nu_1\cdots\nu_m,\mu_1\cdots\mu_n,a_2a_1}(q_1,\ldots,q_m;q_1,\ldots,q_n;s,r)\,, \end{aligned} \tag{6.1}$$

as is clear from figure 6, with again similar identities for the $\alpha\beta$ vertices of (4.4)–(4.6).

6.3 Poincaré invariance

Note that (as usual) all vertices accompany δ functions over the sum of their momentum arguments. The vertices are thus meaningful only when momentum is conserved at the vertex (cf. in particular section 5).

As in ref. [1] we will use the fact that Lorentz invariance implies that changing the sign of all momentum arguments in any vertex, changes the sign of those with an odd number of Lorentz indices and has no effect on those with an even number. (Of course this applies to any even dimension D. In odd dimensions we need also parity, which is a symmetry realised straightforwardly here, to rule out the appearance of $\varepsilon_{\mu_1\cdots\mu_D}$.)

6.4 Dimensions

There is of course a scale invariance corresponding to the naïve, or engineering, dimensions. However for a number of reasons (see sections 2, 3 and 9), especially in general dimension D, the assignments are a little novel. Of course $[S]=0$ but we also have

$$[g^2] = 4-D\,, \qquad [\beta_i] = (D-4)i\,, \qquad [\mathcal{L}_i] = (D-4)i+4\,, \qquad [\hat{\mathcal{L}}] = 4\,,$$

where the $\mathcal{L}_i$ and $\hat{\mathcal{L}}$ are the lagrangians corresponding to S_i and $\hat{S}$. And we have

$$[\mathbf{A}] = [\mathbf{B}] = 1 \qquad \text{and} \qquad [\mathbf{C}] = 0\,.$$

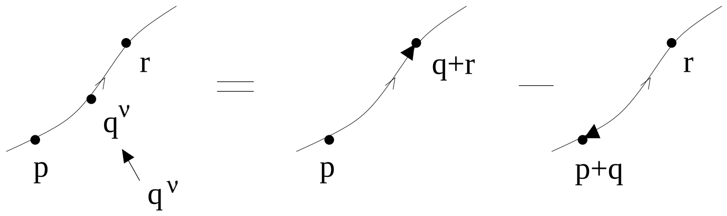


Figure 9: Graphical representation of gauge invariance identities.

6.5 Gauge invariance

These trivial Ward identities follow from the gauge invariance relations (2.3) or from Wilson line representations as in (4.2), just as they did in ref. [1]. They apply to any pure gauge section (i.e. that came from pure $\mathcal{A}$ or pure A) and diagrammatically appear as in figure 9. Thus,

$$q^\nu U^{\cdots \mathbf{XAY} \cdots}_{\cdots a\, \nu\, b \cdots}(\dots, p, q, r, \dots) = U^{\cdots \mathbf{XY} \cdots}_{\cdots a\, b \cdots}(\dots, p, q+r, \dots) - U^{\cdots \mathbf{XY} \cdots}_{\cdots a\, b \cdots}(\dots, p+q, r, \dots)\,, \tag{6.2}$$

where U is some vertex, and a and b are Lorentz indices or null as appropriate. Refs. [1, 3] give as an example the relations for pure gauge (seed) action vertices. For wine vertices where the point is at the end of the line [1]:

$$\begin{aligned} p_1^{\mu_1} W_{\mu_1 \cdots \mu_n, \nu_1 \cdots \nu_m}(p_1, \dots, p_n; q_1, \dots, q_m; r, s) = \\ = W_{\mu_2 \cdots \mu_n, \nu_1 \cdots \nu_m}(p_1+p_2, p_3, \cdots, p_n; q_1, \dots, q_m; r, s) \\ - W_{\mu_2 \cdots \mu_n, \nu_1 \cdots \nu_m}(p_2, \dots, p_n; q_1, \dots, q_m; r+p_1, s)\,, \end{aligned} \tag{6.3}$$

with similar identities for contraction with $p_n^{\mu_n}$, $q_1^{\nu_1}$ and $q_m^{\nu_m}$, as is clear from figure 6. In particular, note the signed momentum that appears for $L_{\mu,\beta}$ of (4.5):

$$p^\mu L_{\mu,\beta}(p; q, r) = L_{q,\beta} - L_{-r,\beta}\,.$$

6.6 Charge conjugation invariance

Recall that the action of charge conjugation on the gauge fields $A^i_\mu \mapsto -{A^i_\mu}^T$, corresponds to reversal of the sign of the underlying Wilson loops [3, 1]. This means that the action is invariant under replacing all supertraces of n gauge fields by the signed reversed order:

$$\mathrm{str}\, \mathbf{A}_{\mu_1} \mathbf{A}_{\mu_2} \cdots \mathbf{A}_{\mu_n} \mapsto (-)^n\, \mathrm{str}\, \mathbf{A}_{\mu_n} \cdots \mathbf{A}_{\mu_2} \mathbf{A}_{\mu_1}\,. \tag{6.4}$$

The ‘peppering’ prescription [3] which as we have seen amounts to replacing $\mathbf{A}$ by $\mathcal{A} = \mathbf{A} + \mathbf{B}$, means that symmetry (6.4) must extend to $\mathbf{B}$ fields also. From the form of $\mathcal{B}$ in (2.24), we then see that $\mathbf{C}$ must also be odd under charge conjugation.

However, to implement this symmetry on the fields, we need to include an extra sign for every pair of B or $\bar{B}$ to compensate for the anticommutation on rearranging the order as in (6.4). Since by fermion number conservation we know there are as many $\bar{B}$s as Bs (see below), we can incorporate this by an extra sign in the definition of the supermatrix transpose; thus we define

$$\mathbf{X}^T := \begin{pmatrix} X^{11\,T} & -X^{21\,T} \\ X^{12\,T} & X^{22\,T} \end{pmatrix}. \tag{6.5}$$

(We could instead place the sign on X^{12}, or an i (or $-i$) in front of both fermionic parts.) The action of charge conjugation symmetry on the fields is then simply summarised as

$$\mathcal{A}_\mu \mapsto -\mathcal{A}_\mu^T, \qquad \mathbf{C} \mapsto -\mathbf{C}^T. \tag{6.6}$$

Note that the extension to $\mathrm{SU}(N|N)$ thus forces charge conjugation to be no longer Z_2, rather it closes on the Z_2 in fermion number conservation: $\mathbf{B} \leftrightarrow -\mathbf{B}$, making a Z_4 in all.

At the 'Wilson loop' level, the symmetry (6.4) thus holds for all the fields $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Of course charge conjugation (6.6) does not act on embedded σ_3s. Instead they maintain their relative position in the Wilson loop, i.e. flip order together with the fields in (6.4) but with $(-)^n$ counting only the fields, as is clear from (6.6). This brings us to an important difference with spontaneously broken $\mathrm{SU}(N|N)$: in the present formulation the embedded σ_3s transform differently from $\mathbf{C}$, and therefore cannot be directly regarded as arising from $\langle\mathcal{C}\rangle$. This difference is entirely due to the 'missing' σ_3 in $\hat{S}_B$ discussed below (2.24), which would replace the lone $\mathbf{B}$ in $\mathcal{B}$ by $i\sigma_3\mathbf{B}$, and thus determine $\mathbf{C}$ to be even under charge conjugation.

6.7 Reality

Note that of course, we also have the requirement of reality of the euclidean action $\hat{S}$ and the flow equation (equivalent, in a time-reversal invariant theory such as this, to unitarity of the Minkowksi theory [15]. Instantons break time-reversal invariance and lead to a complex euclidean S, but the underlying $\hat{S}$ and exact RG equation must still be real. Instanton contributions will be considered in more detail elsewhere.) For the gauge fields, the constraints of reality on the vertices follow after hermitean conjugation and the substitution (or identification) $A^i_\mu \mapsto A^{i\,\dagger}_\mu$. This extends to a change of variables on the superfields thus, $\mathcal{A}_\mu \mapsto \mathcal{A}_\mu^\dagger$ and $\mathbf{C} \mapsto \mathbf{C}^\dagger$. But note that for the same reasons as before, the transpose part must be defined as in (6.5), and thus the superfields are with this definition only "pseudo-real": hermitean conjugation, twice-performed, closes on $\mathbf{B} \leftrightarrow -\mathbf{B}$.

Combining reality, charge conjugation and the comments on Poincaré invariance, one readily shows that in momentum space the (wine or action) vertices associated with an odd number of $\mathbf{C}$ fields (and any number of $\mathbf{A}$s or $\mathbf{B}$s) are pure imaginary, whereas those associated with an even number of $\mathbf{C}$ fields are real. As with the other symmetries outlined in this section, this may be readily verified on the Feynman rules of section 4.

6.8 Fermion number

In any vertex there are always as many Bs as $\bar{B}$s. This has to be so by the remaining $\mathrm{SU}(N)\times\mathrm{SU}(N)$ invariance (cf. section 2 and ref. [3]), but it also implies the existence of a fermion number U(1) symmetry: $B \mapsto B\,\mathrm{e}^{i\vartheta}$ and $\bar{B} \mapsto \bar{B}\,\mathrm{e}^{-i\vartheta}$ (and similarly with B replaced by D, when not eaten). This U(1) is generated by σ_3:

$$\mathbf{X} \mapsto \mathrm{e}^{i\vartheta\sigma_3/2}\,\mathbf{X}\,\mathrm{e}^{-i\vartheta\sigma_3/2}\,, \tag{6.7}$$

where $\mathbf{X}$ runs over the fields, and thus extends the global $\mathrm{SU}(N|N)$ to $\mathrm{U}(N|N)$, appearing in this sense in the usual way. Note however there is of course also an essential difference compared to the usual (i.e. bosonic) groups, in that this extra U(1) acts non-trivially in the adjoint representation. (Also note that the fermion number Z_2 invariance mentioned above and whose explicit representation appears below (2.9), is just the subgroup generated by $\vartheta = \pi$.)

6.9 Duality

While the above symmetries are interesting to explore in their own right and as we will see, are required in practical calculation, by far the most intriguing symmetry we uncovered is a Z_2 duality symmetry which exchanges the rôle of the two groups in $\mathrm{SU}(N)\times\mathrm{SU}(N)$ and at the same time, in a sense that we make explicit below, changes the sign of the squared coupling constant. At the level of the flow equation (2.27), it is implemented by

$$g^2 \mapsto -g^2\,, \qquad \mathcal{A} \mapsto \mathcal{A}^e \qquad \text{and} \qquad \mathbf{C} \mapsto \mathbf{C}^e\,, \tag{6.8}$$

where $\mathbf{X}^e := \sigma_1\mathbf{X}\,\sigma_1$ and thus for example

$$\mathcal{A}^e = \begin{pmatrix} A^2 & \bar{B} \\ B & A^1 \end{pmatrix}. \tag{6.9}$$

(σ_1 was introduced in section 2.) This transformation is of course not part of $\mathrm{U}(N|N)$, not the least because σ_1 has bosonic off-diagonal elements. Note also that $(\sigma_3)^e = -\sigma_3$. Thus from the identity $\mathrm{str}\,\mathbf{X} = \mathrm{tr}\,\sigma_3\mathbf{X}$ (or explicitly from (6.9)) the supertrace of a string of fields is antisymmetric under (6.8). Since from (2.23), $\hat{S}$ is such a single supertrace of superfields, it changes sign. Similarly the r.h.s. of the flow equation (2.27) picks up a sign via the single supertrace in (2.17).[7] Changing also the sign of g^2 in Σ_g of (2.27), we see that the exact RG is indeed invariant. It follows that if $S[\mathbf{A},\mathbf{B},\mathbf{C}](g^2)$ is a solution, then so is $S[\mathbf{A}^e,\mathbf{B}^e,\mathbf{C}^e](-g^2)$.

If we imagine g in (2.27) to be a fixed (i.e. independent of Λ) expansion parameter, for example the classical or bare coupling, $g = g_0$ at $\Lambda = \Lambda_0$, then it is easy to see that we can take S to be self-dual:

$$S[\mathbf{A},\mathbf{B},\mathbf{C}](g_0^2) = S[\mathbf{A}^e,\mathbf{B}^e,\mathbf{C}^e](-g_0^2)\,. \tag{6.10}$$

[7]Note that the change of variables implied by (6.9), and similarly $C^1 \leftrightarrow C^2$, means that the functional derivatives (2.4) are minus their duals.

Indeed this follows immediately if as will be the case, the 'initial' condition, S at $\Lambda = \Lambda_0$, is taken to be self-dual. As a corollary we find from (3.2), that all even (odd) order loop corrections S_n are even (odd) under duality. In the large-N limit, since S is a single supertrace, an even (odd) loop S_n must thus contain an even (odd) number of embedded σ_3s, and this fact can readily be confirmed explicitly by considering figure 2.

The self-duality at the very least is obscured when we come to renormalize however. We see immediately that the β function (3.3) cannot be invariant under (6.8) unless all the odd-loop β_{2n+1} vanish, which is not the case. From the expansion (3.5), (3.6), etc., the non-zero $2\beta_{2n+1}S_0$ terms mix together terms with even and odd numbers of embedded σ_3s, so that the corollary above no longer holds. The underlying reason for these complications is as follows. By the above analysis, at one loop (using (2.29) and solving β)

$$S[\mathbf{A},0,0] = \frac{1}{2}\,\mathrm{str}\int d^Dx \left(\frac{1}{g_0^2} + 2\beta_1\sigma_3 \ln\frac{\Lambda_0}{\Lambda}\right)\mathbf{F}_{\mu\nu}^2 + O(\partial^3)\,. \tag{6.11}$$

Defining $1/g^2 = 1/g_0^2 + 2\beta_1 \ln(\Lambda_0/\Lambda)$ only absorbs the divergence for the A^1 part of the action. We see that in the continuum limit we are forced to introduce two renormalised couplings: $g = g_1$ for A^1, and g_2 for A^2. Under duality we map to a solution of (2.27), for which the renormalization condition (2.29) now insists $g = g_2$. Let us call the couplings for such a solution $\tilde{g}_2 = \tilde{g}$ and $\tilde{g}_1$. Then by the duality of (2.27) and the initial bare action,

$$\begin{aligned}\tilde{g}_2^2\left(g_0^2,\frac{\Lambda_0}{\Lambda}\right) &= -g^2\left(-g_0^2,\frac{\Lambda_0}{\Lambda}\right) g_2^2\left(g_0^2,\frac{\Lambda_0}{\Lambda}\right)\\ &= -\tilde{g}_1^2\left(-g_0^2,\frac{\Lambda_0}{\Lambda}\right).\end{aligned} \tag{6.12}$$

We comment further in the conclusions.

7. Classical vertices without gauge fixing

As we will see, even here there are surprises. *Classical* solutions will turn out to suffer a form of divergence, arising from integration over Λ, which is regularised by careful choice of the $\Lambda = \Lambda_0 \to \infty$ 'boundary conditions', and will require the introduction of some new 'renormalised' parameters [3]. This has nothing to do with unbounded momentum integrals, and nothing to do with gauge invariance *per se*: it arises in the Pauli-Villars sector from the existence of positive powers of the cutoff and some freedom to add extra interactions. However, the implementation of a Pauli-Villars exact RG scheme is in itself one of the novel developments we report in this paper.

The dictionary for translating the Feynman diagrams, which themselves follow from expanding (the relevant parts of) figure 1, has already been given in section 4. In fact the expanded Wilson loops look identical to those in refs. [1, 3], a consequence of the equality of form with that of the pure gauge case of both figure 1 and the perturbative development in section 3.

As already noted in section 2, the tree-level insertions of σ_3 in (2.11) serve only to ensure that $\mathbf{Y}$ and $\mathbf{X}$, the remainders of the Feynman diagram on either side, are bosonic or fermionic as appropriate.[8] But these restrictions are automatically incorporated in the explicit Feynman rules, cf. section 4. Constructing the tree-level vertices out of them, expanding using figure 2 and (anti)commuting the σ_3s together, they all combine to no overall effect. Thus we omit this step in this section.

Solving the flow equations for the vertices introduces integration constants, i.e. terms independent of Λ. Of course these must be chosen to satisfy all the symmetries of the theory, and we will signal which symmetries provide non-trivial constraints. Moreover at the classical level, since g does not run, there is no difficulty in preserving the self-duality (6.10), which follows here because the integration constants will be chosen to be single supertraces without embedded σ_3s. Thus in this formulation, there are no classical vertices with an embedded σ_3:

$$S^{0\sigma\mathbf{X}_1\cdots\mathbf{X}_n}_{\;a_1\cdots a_n} = 0\,. \tag{7.1}$$

The reader can find in section 6, the list of symmetries together with relevant comments and definitions.

It is helpful also, to borrow the conclusions on 'drifting' from section 9, in particular those summarised in lemmas 3 and 4 and corollary 5. Thus we already know that pure-$\mathbf{A}$ classical vertices are constructed only out of $\hat{S}_A$ vertices, such as in (4.11), and c' vertices i.e. the non-L part of $c'_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m,\alpha\beta}$ (cf. (4.4)). In other words, the pure-$\mathbf{A}$ S_0 vertices are unchanged from ref. [1]. Similarly to (4.10), let us reserve the unlabelled $S^0_{\mu_1\cdots\mu_n}$ vertices for these, which thus have the same explicit expressions as in refs. [1, 3]. We also know there are no S_0 vertices with just one $\mathbf{C}$.

Contributions that seem at variance with the above conclusions, vanish as a consequence of drifting, which itself is a consequence of the exact preservation of gauge invariance. Although these statements are readily verified, in the interests of compactness we omit the explicit computations.

7.1 Two point vertices

From (3.4), we thus have [1, 3] figure 10.

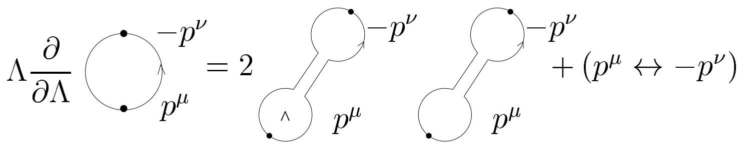


Figure 10: Feynman diagrams for the two-point vertex.

[8]More strictly block diagonal or block off-diagonal as appropriate.

As in refs [3, 1], we now adopt the convention that the empty circle corresponds to S_0, not S as in figure 1, and we have noted that once again since actions' one-point vertices vanish (for example by charge conjugation invariance) we must have at least one blob per lobe. Here however, we must also assign flavours **A**, **B** or **C** to the two points.

From (4.12) and (4.7), at first sight we appear to generate a mixed **C**-**A** vertex. Actually, the required **C**-**A** zero-point wine vertex anihilates the **A**-**A** lobe by gauge invariance: $p^\mu \hat{S}_{\mu\nu}(p) = 0$. This is nothing but a perturbative verification of the consequences of 'drifting' as discussed above. It is consistent to set to zero all S_0 vertices for which the r.h.s. of the flow equation vanishes, as here, and we will do so in this paper. (In fact in this case, it may be verified that the requirements of gauge invariance and Lorentz invariance already disallow a **C**-**A** vertex.)

We are left with:

$$\Lambda \frac{\partial}{\partial \Lambda} S^0_{\mu\nu}(p) = \frac{1}{2\Lambda^2} c'_p \left[2\hat{S}_{\mu\alpha}(p) - S^0_{\mu\alpha}(p) \right] S^0_{\alpha\nu}(p) + (p_\mu \leftrightarrow -p_\nu)\,, \tag{7.2}$$

$$\Lambda \frac{\partial}{\partial \Lambda} S^{0\mathbf{BB}}_{\;\mu\,\nu}(p) = \frac{1}{2\Lambda^2} \left[2\hat{S}^{\mathbf{BB}}_{\mu\,\alpha}(p) - S^{0\mathbf{BB}}_{\;\mu\,\alpha}(p) \right] K_{p,\alpha\beta}\, S^{0\mathbf{BB}}_{\;\beta\,\nu}(p) + (p_\mu \leftrightarrow -p_\nu)\,, \tag{7.3}$$

$$\Lambda \frac{\partial}{\partial \Lambda} S^{0\mathbf{CC}}(p) = \frac{1}{\Lambda^4} M_p \left[2\hat{S}^{\mathbf{CC}}(p) - S^{0\mathbf{CC}}(p) \right] S^{0\mathbf{CC}}(p)\,, \tag{7.4}$$

where we have used (4.7), and on (7.2), the drifting simplifications mentioned above.

For completeness we recall how (7.2) is solved [1]. By gauge invariance and dimensions,

$$S^0_{\mu\nu}(p) = \frac{2\Delta_{\mu\nu}(p)}{f(p^2/\Lambda^2)}\,. \tag{7.5}$$

From (3.2), we require $f(0) = 1$ so as to be consistent with (2.29) in the $g \to 0$ limit. Substituting (4.11), one readily finds the unique solution to be $f = c$.

Similarly, substituting (4.12), (4.5), (4.4) and (2.26), one readily verifies that the two-point classical and seed vertices for **B** and **C** may also be taken to be equal. Thus in total:

$$S^0_{\mu\nu}(p) = \hat{S}_{\mu\nu}(p)\,, \qquad S^{0\mathbf{BB}}_{\;\mu\,\nu}(p) = \hat{S}^{\mathbf{BB}}_{\mu\,\nu}(p)\,, \qquad S^{0\mathbf{CC}}(p) = \hat{S}^{\mathbf{CC}}(p)\,. \tag{7.6}$$

In the case of **B** and **C** however, these are not the most general solutions, presumably reflecting the freedom of reparametrization invariance in the non-gauge sector [7, 16], but we will specialize to these equalities since they simplify the higher-point vertices as in ref. [1]. In fact the expressions for K, L and M in (2.26) were determined to make these equalities possible. (See also section 9.)

7.2 Three-point vertices

Similarly, from (3.4) and the top two lines of figure 1, we obtain for the three-point vertex the diagrams in figure 11.

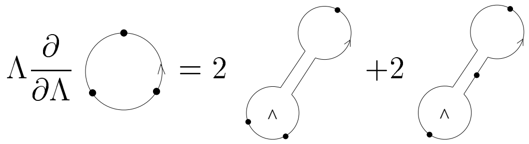


Figure 11: Feynman diagrams for the three-point vertex.

Here we have already simplified with (7.6), which works precisely in the same way as in ref. [1]. We are thus again [1] left on the r.h.s. with terms that are already determined, allowing the differential equation to be integrated immediately. Again this simplification, provided by the equalities in (7.6), persists to all higher point S_0 vertices. The proof is identical to that in ref. [1].

Up to cyclicity, the possible three-point flavours are **AAA**, **BBA**, **BBC**, **CCA**, and **CCC**. Note that odd numbers of **B**s are ruled out by fermion number (the supertrace of such an odd number vanishes identically) and the classical **CAA** vertex is ruled out by drifting (cf. discussion above).

For the pure **A** vertex, all the L terms vanish by drifting (again as outlined above) and we thus obtain the same flow equation as in ref. [1], with the same result:

$$\begin{aligned} S^0_{\mu\nu\lambda}(p,q,r) = &-\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1^3}\left\{c'_r \hat{S}_{\mu\nu\alpha}(p,q,r)\hat{S}_{\alpha\lambda}(r) + c'_\nu(q;p,r)\hat{S}_{\mu\alpha}(p)\hat{S}_{\alpha\lambda}(r)\right\} + \\ &+ 2(r_\nu\delta_{\mu\lambda} - r_\mu\delta_{\nu\lambda}) + \text{cycles}\,. \end{aligned} \tag{7.7}$$

As in refs. [1, 3], we adopt the convention that the terms in a particular Λ-integral are understood to have Λ replaced by the integration variable (here Λ_1). The term "cycles" means that we add to the expression the two cyclic permutations of $(p_\mu, q_\nu, r_\lambda)$. Recall from ref. [1] (see also ref. [3]), that the continuum limit, corresponding to the upper limit $\Lambda_1 = \infty$, trivially exists, and that the integration constant is fixed by gauge invariance to be the unique covariantization of $\Delta_{\mu\nu}$, i.e. the usual 'bare' three-point vertex, as is also clear from the $\Lambda \to \infty$ limit of (7.7), since from (2.29), and (6.10), and by dimensions

$$S_0[\mathbf{A},0,0] = \frac{1}{2}\,\mathrm{str}\int d^Dx\,\mathbf{F}^2_{\mu\nu} + O(\partial^3/\Lambda) \tag{7.8}$$

(or simply by restriction to A^1).

In a similar way we obtain for the **BBA** vertex:

$$\begin{aligned} S^{0\mathbf{BBA}}_{\mu\nu\lambda}(p,q,r) = -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1^3}\Big\{&\hat{S}^{\mathbf{BBA}}_{\mu\nu\alpha}(p,q,r)c'_r\hat{S}_{\alpha\lambda}(r) + \hat{S}^{\mathbf{BBA}}_{\mu\alpha\lambda}(p,q,r)K_{q,\alpha\beta}\hat{S}^{\mathbf{BB}}_{\beta\nu}(q) + \\ &+\hat{S}^{\mathbf{BBA}}_{\alpha\nu\lambda}(p,q,r)K_{p,\alpha\beta}\hat{S}^{\mathbf{BB}}_{\beta\mu}(p) + \hat{S}^{\mathbf{BB}}_{\mu\alpha}(p)c'_{\nu,\alpha\beta}(q;p,r)\hat{S}_{\beta\lambda}(r) + \\ &+\hat{S}_{\lambda\alpha}(r)c'_{\mu,\alpha\beta}(p;r,q)\hat{S}^{\mathbf{BB}}_{\beta\nu}(q) + \hat{S}^{\mathbf{BB}}_{\nu\alpha}(q)K_{\lambda,\alpha\beta}(r;q,p)\hat{S}^{\mathbf{BB}}_{\beta\mu}(p)\Big\} + \\ +2\Big\{&(r_\nu\delta_{\mu\lambda} - r_\mu\delta_{\nu\lambda})\left(1+\frac{\gamma^{BBA}}{2}\right) + p_\lambda\delta_{\nu\mu} - p_\nu\delta_{\lambda\mu} + \\ &+ q_\mu\delta_{\lambda\nu} - q_\lambda\delta_{\mu\nu} + \frac{\tilde{c}'_0}{\tilde{c}_0^2}\delta_{\mu\nu}(q-p)_\lambda\Big\}\,, \end{aligned} \tag{7.9}$$

where γ^{BBA} is a dimensionless real free parameter, and the integral needs interpreting with some care — as we explain below. The first term has been simplified, once again by drifting — i.e. gauge invariance (as above (7.2)–(7.4)). In fact in this way, expanding the wine vertices by (4.4), all the terms in (7.9) containing L, either vanish or considerably simplify. We omit the details.

To justify (7.9), we replace the top limit in the Λ_1 integral by Λ_0, in which case clearly the integration constant is identified with the 'bare' value of $S^{0\mathbf{BBA}}_{\mu\nu\lambda}(p,q,r)$, i.e. its value at $\Lambda=\Lambda_0$. Unlike the pure-$\mathbf{A}$ case viz. (2.29), we do not have a renormalization condition to fix this. (Actually, after identifying all relevant and marginal directions, these could be introduced of course, but as part of the regulating structure they are not needed — nor do they simplify the calculations — at least in this paper.) However by quasilocality [1] and dimensions, since we may discard all terms that vanish in the limit $\Lambda_0\to\infty$, finding the most general integration constant reduces to looking for the usual bare-action type terms[9] consistent with the symmetries of the theory. The exact preservation of $\mathrm{SU}(N)\times\mathrm{SU}(N)$ gauge invariance makes this process simple and elegant.

Thus the most general integration constant is the $\mathbf{BBA}$ part of

$$\mathcal{L}_0|_{\Lambda=\Lambda_0} = \mathrm{str}\left\{\frac{1}{2}\mathcal{F}^2_{\mu\nu} + \frac{\tilde{c}'_0}{\tilde{c}_0^2}\mathbf{B}_\mu\nabla^2\mathbf{B}_\mu + i\gamma^{BBA}\mathbf{B}_\mu\mathbf{F}_{\mu\nu}\mathbf{B}_\nu + \cdots\right\}, \tag{7.10}$$

where the first two terms are fixed by gauge invariance from (7.6).[10] The ellipses refer to gauge invariant terms that do not contain the $\mathbf{BBA}$ vertex (and note by (2.23), include some that diverge in the $\Lambda\to\infty$ limit). Note that the last term has the right reality and charge conjugation properties.

As we will verify, the universality of the continuum limit means that physical quantities will be independent of γ^{BBA}. But we cannot quite just set it to zero: by dimensions we see that if the integral in (7.9) yields such a term, then it is logarithmically divergent (viz. $\sim\int_\Lambda^{\Lambda_0} d\Lambda_1/\Lambda_1$). However, also note that since (7.10) contains the most general possible such Λ-independent terms, the Λ_1 integral can *only* diverge this way. A straightforward calculation confirms that the integral in (7.9) indeed diverges, as $-16\ln(\Lambda_0/\Lambda)\,(r_\nu\delta_{\mu\lambda}-r_\mu\delta_{\nu\lambda})$, and thus for a finite continuum limit we set the constant to

$$\gamma^{BBA} = 16\ln\left(\frac{\Lambda_0}{\mu}\right) + \text{finite}\,, \tag{7.11}$$

where μ is a finite mass scale, and $\Lambda_0\to\infty$.

At a more sophisticated level, we may simply impose a definite prescription for discarding the infinities in Λ integrals arising in finite continuum solutions such

[9] I.e. polynomial in momenta, balanced as required by non-negative powers of Λ_0.

[10] Viz. gauge covariantizing (4.12), or from (2.23), or directly by expanding (4.13); here and later the solution may also be readily derived directly in momentum space, using (6.2).

as (7.9), for example minimal subtraction of Λ_0 divergences, safe in the knowledge that in reality these divergences are actually cancelled by opposite divergences in parameters in the most general integration constant. We could go further with this prescription, and discard these parameters, since this just amounts to choosing them to be precisely the opposing divergences. We will keep them however, as an extra test of universality, but for simplicity report from now on this more sophisticated approach. (Actually, we also checked the calculations the dumb way, as in (7.11) and above. We omit the details.)

In this way, the **BBC** vertex is found to be

$$\begin{aligned} S^{0\mathbf{BBC}}_{\mu\nu}(p,q,r) = -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1^3}\Big\{ & \frac{1}{\Lambda_1^2}\hat{S}^{\mathbf{BBC}}_{\mu\nu}(p,q,r)M_r\hat{S}^{\mathbf{CC}}(r) + \hat{S}^{\mathbf{BBA}}_{\mu\nu\alpha}(p,q,r)L_{r,\alpha}\hat{S}^{\mathbf{CC}}(r) + \\ & + \hat{S}^{\mathbf{BBC}}_{\mu\alpha}(p,q,r)K_{q,\alpha\beta}\hat{S}^{\mathbf{BB}}_{\beta\nu}(q) + \hat{S}^{\mathbf{BBC}}_{\alpha\nu}(p,q,r)K_{p,\alpha\beta}\hat{S}^{\mathbf{BB}}_{\beta\mu}(p) - \\ & - \hat{S}^{\mathbf{BB}}_{\mu\alpha}(p)L_{\nu,\alpha}(q;r,p)\hat{S}^{\mathbf{CC}}(r) + \hat{S}^{\mathbf{CC}}(r)L_{\mu,\beta}(p;r,q)\hat{S}^{\mathbf{BB}}_{\beta\nu}(q)\Big\} + \\ & + i\gamma_1^{BBC}\delta_{\mu\nu}(p^2-q^2) + i\gamma_2^{BBC}(p_\mu p_\nu - q_\mu q_\nu)\,, \end{aligned} \tag{7.12}$$

where the two dimensionless real γ_i^{BBC} parametrise the most general integration constant. Once again, the L parts considerably simplify on using the gauge invariance relations (6.2). Note that charge conjugation invariance requires the integration constants to be odd under $p_\mu \leftrightarrow q_\nu$. Dimensions suggest and explicit calculation confirms that the γ_i^{BBC} mop up logarithmic divergences in the Λ_1 integral.

The **CCA** vertex is found to be:

$$\begin{aligned} S^{0\mathbf{CCA}}_{\lambda}(p,q,r) = -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1^5}\Big\{ & \hat{S}^{\mathbf{CCA}}_{\lambda}(p,q,r)\Big[M_pS^{\mathbf{CC}}(p) + M_qS^{\mathbf{CC}}(q)\Big] + \\ & + \Lambda_1^2\hat{S}^{\mathbf{CCA}}_{\alpha}(p,q,r)c'_r\hat{S}_{\alpha\lambda}(r) + \hat{S}^{\mathbf{CC}}(q)M_\lambda(r;q,p)\hat{S}^{\mathbf{CC}}(p)\Big\} + \\ & + 2\frac{\tilde{c}'_0}{\tilde{c}_0^2}(p^2+q^2)(q-p)_\lambda + \gamma^{CCA}(r^2p_\lambda - r_\lambda r\cdot p)\,, \end{aligned} \tag{7.13}$$

where γ^{CCA} is another dimensionless real free parameter. Here the integration constant is constrained by gauge invariance, (7.6) and (2.23) to be the **CCA** vertex in

$$\mathcal{L}_0|_{\Lambda=\Lambda_0} = \mathrm{str}\,\nabla_\mu\cdot\mathbf{C}\left(\Lambda_0^2\delta_{\mu\nu} + \frac{\tilde{c}'_0}{\tilde{c}_0^2}\delta_{\mu\nu}\nabla^2 + i\gamma^{CCA}\mathbf{F}_{\mu\nu}\right)\nabla_\nu\cdot\mathbf{C} + \cdots, \tag{7.14}$$

where the ellipses do not contain **CCA** vertices. The first term precisely cancels an equal and opposite quadratic divergence in the Λ_1 integral, as it must, and thus by the prescription below (7.11), is not displayed in (7.13). The last term, which verifies the reality and charge conjugation symmetries, mops up a logarithmic divergence in the Λ_1 integral.

Finally, the three-point classical **C** vertex is:

$$S^{0\mathbf{CCC}}(p,q,r) = i\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1^5}\left\{\left[\hat{S}^{\mathbf{CC}}(p) - \hat{S}^{\mathbf{CC}}(q)\right]L_r\hat{S}^{\mathbf{CC}}(r)\ + \text{cycles}\right\}, \tag{7.15}$$

where 'cycles' stands for the two cyclic permutations of (p, q, r). Here we have 'drifted' $\hat{S}^{\mathbf{CCA}}_{\lambda} L_{r,\lambda}$ terms by using (4.5) and (6.2). Remarkably, the integration constant must vanish. Charge conjugation and cyclic symmetry require that it be antisymmetric under exchange of any pair of momenta. In this case, any such expression can be generated by writing down Lorentz invariant terms antisymmetric under $p \leftrightarrow q$ and then adding the cyclic iterands. Using momentum conservation, it is then straightforward to show that all such polynomials up to dimension 4 vanish on adding the cycles. Similarly one checks that the superficially quartically divergent integral in (7.15) is actually finite, as consistency requires.

7.3 Four-point vertices

As already discussed, the diagrams are the same as in ref. [1] (figure 12).

We will concentrate on the vertices that we will need for the β_1 computation: **AAAA**, **BBAA** and **CCAA**. (All the three-point S_0 vertices are however needed, either to derive these four-point vertices or directly.) The **CAAA** vertex need not be considered since it vanishes by drifting.

For the same reasons as given for the three-point vertex (7.7), we obtain the same four-**A** vertex as in ref. [1]:

$$
\begin{aligned}
S^0_{\mu\nu\lambda\sigma}(p,q,r,s) = -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1^3}\Big\{ & c'_{p+q}\Big(\hat{S}_{\mu\nu\alpha}(p,q,r+s) - \\
& - \frac{1}{2}S^0_{\mu\nu\alpha}(p,q,r+s)\Big) S^0_{\alpha\lambda\sigma}(p+q,r,s) + \\
& + \hat{S}_{\sigma\alpha}(s)\hat{S}_{\mu\nu\alpha}(p,q,r+s) c'_\lambda(r;p+q,s) + \\
& + \hat{S}_{\lambda\alpha}(r)\hat{S}_{\mu\nu\alpha}(p,q,r+s) c'_\sigma(s;r,p+q) + \\
& + \hat{S}_{\mu\alpha}(p)\hat{S}_{\alpha\sigma}(s) c'_{\nu\lambda}(q,r;p,s) + \\
& + \frac{1}{2}\hat{S}_{\mu\alpha}(p)\hat{S}_{\alpha\lambda}(r) c'_{\nu,\sigma}(q;s;p,r) + \\
& + c'_s \hat{S}_{\sigma\alpha}(s)\hat{S}_{\mu\nu\lambda\alpha}(p,q,r,s) + \text{cycles}\Big\} + \\
& + 2\delta_{\mu\sigma}\delta_{\nu\lambda} - 4\delta_{\mu\lambda}\delta_{\nu\sigma} + 2\delta_{\mu\nu}\delta_{\lambda\sigma}\,, \qquad (7.16)
\end{aligned}
$$

where 'cycles' stands for the three cyclic permutations of $(p_\mu, q_\nu, r_\lambda, s_\sigma)$.

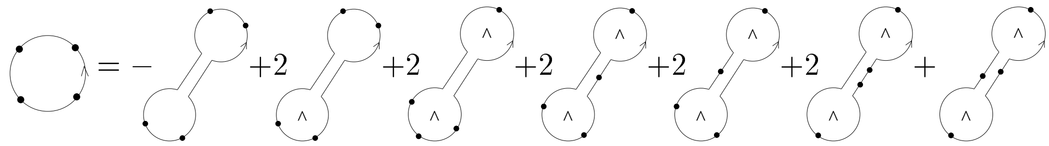
$$\Lambda \frac{\partial}{\partial \Lambda}$$ 

Figure 12: Feynman diagrams for the four-point vertex.

The **BBAA** vertex is:

$$
\begin{aligned}
&S^{0\mathbf{BBAA}}_{\mu\nu\lambda\sigma}(p,q,r,s) = \\
&= -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1^3}\Big\{ - S^{0\mathbf{BBA}}_{\mu\nu\alpha}(p,q,r{+}s)c'_{p+q}S^0_{\alpha\lambda\sigma}(p{+}q,r,s) - \\
&\qquad - S^{0\mathbf{BBA}}_{\mu\alpha\sigma}(p,q{+}r,s)K_{p+s,\alpha\beta}S^{0\mathbf{BBA}}_{\beta\nu\lambda}(p{+}s,q,r) + \\
&\qquad + \hat{S}^{\mathbf{BBA}}_{\mu\nu\alpha}(p,q,r{+}s)c'_{p+q}S^0_{\alpha\lambda\sigma}(p{+}q,r,s) + \\
&\qquad + S^{0\mathbf{BBA}}_{\mu\nu\alpha}(p,q,r{+}s)c'_{p+q}\hat{S}_{\alpha\lambda\sigma}(p{+}q,r,s) + \\
&\qquad + \hat{S}^{\mathbf{BBA}}_{\mu\alpha\sigma}(p,q{+}r,s)K_{p+s,\alpha\beta}S^{0\mathbf{BBA}}_{\beta\nu\lambda}(p{+}s,q,r) + \\
&\qquad + S^{0\mathbf{BBA}}_{\mu\alpha\sigma}(p,q{+}r,s)K_{p+s,\alpha\beta}\hat{S}^{\mathbf{BBA}}_{\beta\nu\lambda}(p{+}s,q,r) + \\
&\qquad + \hat{S}^{\mathbf{BBAA}}_{\mu\nu\lambda\alpha}(p,q,r,s)c'_s\hat{S}_{\alpha\sigma}(s) + \hat{S}^{\mathbf{BBAA}}_{\mu\nu\alpha\sigma}(p,q,r,s)c'_r\hat{S}_{\alpha\lambda}(r) + \\
&\qquad + \hat{S}^{\mathbf{BBAA}}_{\mu\alpha\lambda\sigma}(p,q,r,s)K_{q,\alpha\beta}\hat{S}_{\beta\nu}(q) + \hat{S}^{\mathbf{BBAA}}_{\alpha\nu\lambda\sigma}(p,q,r,s)K_{p,\alpha\beta}\hat{S}_{\beta\mu}(p) + \\
&\qquad + \hat{S}^{\mathbf{BBA}}_{\mu\nu\alpha}(p,q,r{+}s)c'_\lambda(r;p{+}q,s)\hat{S}_{\alpha\sigma}(s) + \\
&\qquad + \hat{S}^{\mathbf{BBA}}_{\mu\alpha\sigma}(p,q{+}r,s)c'_{\nu,\alpha\beta}(q;p{+}s,r)\hat{S}_{\beta\lambda}(r) + \\
&\qquad + \hat{S}_{\lambda\sigma\alpha}(r,s,p{+}q)c'_{\mu,\alpha\beta}(p;r{+}s,q)\hat{S}^{\mathbf{BB}}_{\beta\nu}(q) + \\
&\qquad + \hat{S}^{\mathbf{BBA}}_{\alpha\nu\lambda}(p{+}s,q,r)K_{\sigma,\alpha\beta}(s;q{+}r,p)\hat{S}^{\mathbf{BB}}_{\beta\mu}(p) + \\
&\qquad + \hat{S}^{\mathbf{BBA}}_{\mu\nu\alpha}(p,q,r{+}s)c'_\sigma(s;r,p{+}q)\hat{S}_{\alpha\lambda}(r) + \\
&\qquad + \hat{S}^{\mathbf{BBA}}_{\mu\alpha\sigma}(p,q{+}r,s)K_{\lambda,\beta\alpha}(r;q,p{+}s)\hat{S}^{\mathbf{BB}}_{\beta\nu}(q) + \\
&\qquad + \hat{S}_{\lambda\sigma\alpha}(r,s,p{+}q)c'_{\nu,\beta\alpha}(q;p,r{+}s)\hat{S}^{\mathbf{BB}}_{\beta\mu}(p) + \\
&\qquad + \hat{S}^{\mathbf{BBA}}_{\alpha\nu\lambda}(p{+}s,q,r)c'_{\mu,\beta\alpha}(p;s,q{+}r)\hat{S}_{\beta\sigma}(s) + \\
&\qquad + \hat{S}^{\mathbf{BB}}_{\mu\alpha}(p)c'_{\nu\lambda,\alpha\beta}(q,r;p,s)\hat{S}_{\beta\sigma}(s) + \hat{S}_{\sigma\alpha}(s)K_{\mu\nu,\alpha\beta}(p,q;s,r)\hat{S}_{\beta\lambda}(r) + \\
&\qquad + \hat{S}_{\lambda\alpha}(r)c'_{\sigma\mu,\alpha\beta}(s,p;r,q)\hat{S}^{\mathbf{BB}}_{\beta\nu}(q) + \hat{S}^{\mathbf{BB}}_{\nu\alpha}(q)c'_{\lambda\sigma,\alpha\beta}(r,s;q,p)\hat{S}^{\mathbf{BB}}_{\beta\mu}(p) + \\
&\qquad + \hat{S}^{\mathbf{BB}}_{\mu\alpha}(p)c'_{\nu,\sigma,\alpha\beta}(q;s;p,r)\hat{S}_{\beta\lambda}(r) + \hat{S}^{\mathbf{BB}}_{\nu\alpha}(q)c'_{\lambda,\mu,\alpha\beta}(r;p;q,s)\hat{S}_{\beta\sigma}(s)\Big\} + \\
&\quad + 2\delta_{\mu\sigma}\delta_{\nu\lambda} - 4\delta_{\mu\lambda}\delta_{\nu\sigma} + 2\delta_{\mu\nu}\delta_{\lambda\sigma} - 2\frac{\tilde{c}'_0}{\tilde{c}_0^2}\delta_{\mu\nu}\delta_{\lambda\sigma} + \gamma^{BBA}(\delta_{\mu\sigma}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\sigma}). \quad (7.17)
\end{aligned}
$$

The integrand is a straightforward translation from the diagrams, together with some cancellations of L wine terms, in particular all **C**-**A** wine terms, as a result of drifting. Further such simplifications occur only at the expense of expanding the seed vertices containing **B**, as $\hat{S}_A + \hat{S}_B$, to isolate the pure gauge part $\hat{S}_A$. Note that, as with the four-**A** vertex [1], gauge invariance (6.2) strongly constrains the overall form of the integrand and acts as a powerful consistency check. The integration constant is dimension zero and therefore cannot generate any new gauge invariant terms. Consequently the integration constant follows from expansion of (7.10), and the Λ_1 integral diverges in just such a way as to be cancelled by the divergence in (7.11).

By drifting, all the L wine terms cancel out in the **CCAA** vertex:

$$
\begin{aligned}
&S^{0\mathbf{CCAA}}_{\lambda\sigma}(p,q,r,s) = \\
&= -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1^3}\Bigg\{ -S^{0\mathbf{CCA}}_{\alpha}(p,q,r{+}s)c'_{p+q}S^0_{\alpha\lambda\sigma}(p{+}q,r,s) - \\
&\qquad - \frac{1}{\Lambda_1^2}S^{0\mathbf{CCA}}_{\sigma}(p,q{+}r,s)M_{p+s}S^{0\mathbf{CCA}}_{\lambda}(p{+}s,q,r) + \\
&\qquad + \hat{S}^{\mathbf{CCA}}_{\alpha}(p,q,r{+}s)c'_{p+q}S^0_{\alpha\lambda\sigma}(p{+}q,r,s) + \\
&\qquad + S^{0\mathbf{CCA}}_{\alpha}(p,q,r{+}s)c'_{p+q}\hat{S}_{\alpha\lambda\sigma}(p{+}q,r,s) + \\
&\qquad + \frac{1}{\Lambda_1^2}\hat{S}^{\mathbf{CCA}}_{\sigma}(p,q{+}r,s)M_{p+s}S^{0\mathbf{CCA}}_{\lambda}(p{+}s,q,r) + \\
&\qquad + \frac{1}{\Lambda_1^2}S^{0\mathbf{CCA}}_{\sigma}(p,q{+}r,s)M_{p+s}\hat{S}^{\mathbf{CCA}}_{\lambda}(p{+}s,q,r) + \\
&\qquad + \hat{S}^{\mathbf{CCAA}}_{\lambda\alpha}(p,q,r,s)c'_s\hat{S}_{\alpha\sigma}(s) + \hat{S}^{\mathbf{CCAA}}_{\alpha\sigma}(p,q,r,s)c'_r\hat{S}_{\alpha\lambda}(r) + \\
&\qquad + \frac{1}{\Lambda_1^2}\hat{S}^{\mathbf{CCAA}}_{\lambda\sigma}(p,q,r,s)\left[M_q\hat{S}^{\mathbf{CC}}(q) + M_p\hat{S}^{\mathbf{CC}}(p)\right] + \\
&\qquad + \hat{S}^{\mathbf{CCA}}_{\alpha}(p,q,r{+}s)c'_\lambda(r;p{+}q,s)\hat{S}_{\alpha\sigma}(s) + \\
&\qquad + \frac{1}{\Lambda_1^2}\hat{S}^{\mathbf{CCA}}_{\lambda}(s{+}p,q,r)M_\sigma(s;q{+}r,p)\hat{S}^{\mathbf{CC}}(p) + \\
&\qquad + \hat{S}^{\mathbf{CCA}}_{\alpha}(p,q,r{+}s)c'_\sigma(s;r,p{+}q)\hat{S}_{\alpha\lambda}(r) + \\
&\qquad + \frac{1}{\Lambda_1^2}\hat{S}^{\mathbf{CCA}}_{\sigma}(p,q{+}r,s)M_\lambda(r;q,p{+}s)\hat{S}^{\mathbf{CC}}(q) + \\
&\qquad + \frac{1}{\Lambda_1^2}\hat{S}^{\mathbf{CC}}(q)M_{\lambda\sigma}(r,s;q,p)\hat{S}^{\mathbf{CC}}(p)\Bigg\} + \\
&\qquad + 2\frac{\tilde{c}'_0}{\tilde{c}_0^2}\left\{(2q+r)_\lambda(2p+s)_\sigma - (p^2+q^2)\delta_{\lambda\sigma}\right\} + \\
&\qquad + \gamma^{CCA}\left\{p_\lambda q_\sigma - p_\sigma q_\lambda - \delta_{\lambda\sigma}p.s + s_\lambda p_\sigma - \delta_{\lambda\sigma}q\cdot r + r_\sigma q_\lambda\right\} + \\
&\qquad + \gamma^{CCAA}(\delta_{\lambda\sigma}r\cdot s - s_\lambda r_\sigma)\,.
\end{aligned}
\tag{7.18}
$$

Gauge invariance again acts as a powerful consistency check on the integrand. The dimension two integration constant allows for one new gauge invariant term $\sim \gamma^{CCAA}\mathbf{C}\mathbf{F}^2_{\mu\nu}\mathbf{C}$, otherwise gauge invariance requires the rest to be taken from (7.14). The quadratic divergence $2\Lambda_0^2\tilde{c}_0^{-1}\delta_{\lambda\sigma}$ is not displayed but cancels exactly such a divergence in the integral. Together with γ^{CCA}, our final free real dimensionless parameter γ^{CCAA}, mop up all logarithmic divergences in the integral.

8. The β function without gauge fixing

The β function is determined in essentially the same way as in ref. [1]. As for $\hat{S}$ and S_0, when the vertices are not labelled by their flavours, we now mean the pure-**A** vertices. Using the renormalization condition (2.29), we have for the **AA** vertices

$$S_{\mu\nu}(p) + S^{\sigma}_{\mu\nu}(p) = \frac{2}{g^2}\Delta_{\mu\nu}(p) + O(p^3)\,, \tag{8.1}$$

and thus by (7.6) and (4.11),

$$S_{\mu\nu}(p) + S^{\sigma}_{\mu\nu}(p) = \frac{1}{g^2}S^0_{\mu\nu}(p) + O(p^3)\,.$$

By (3.2) and (7.1), this implies *that the $O(p^2)$ component of all the higher loop contributions $S^n_{\mu\nu}(p) + S^{n\sigma}_{\mu\nu}(p)$, must vanish.* This greatly simplifies the $O(p^2)$ part of the **AA** vertex flow in (3.5)–(3.6), in particular reducing them to algebraic equations. Thus we see that

$$a_1[S_0 - 2\hat{S}]^{\sigma}_{\mu\nu}(p) = -4\beta_1\Delta_{\mu\nu}(p) + O(p^3)\,, \tag{8.2}$$

where $a_1[S_0 - 2\hat{S}]^{\sigma}_{\mu\nu}(p)$ is the σ_3**AA** vertex in $a_1[S_0 - 2\hat{S}]$ and we have used the fact that all one-loop vertices contain a trapped σ_3 (cf. discussion above (6.11)). This fixes β_1. Similarly, at $n \geq 2$ loops, the β_n are determined by the requirement that

$$a_1[S_{n-1}]_{\mu\nu}(p) + a_1[S_{n-1}]^{\sigma}_{\mu\nu}(p) = -4\beta_n\Delta_{\mu\nu}(p) + O(p^3)\,.$$

And non-perturbatively from (3.1) and (3.2),

$$a_1[g^2S - 2\hat{S}]_{\mu\nu}(p) + a_1[g^2S - 2\hat{S}]^{\sigma}_{\mu\nu}(p) = -\frac{4}{g^3}\beta(g)\Delta_{\mu\nu}(p) + O(p^3)\,.$$

8.1 One loop β function without gauge fixing

The diagrams giving the l.h.s. of (8.2) take the same form as in ref. [1], figure 13 (apart from the fact that the large-N limit has been taken).

where, as in ref.[1], we let the circle stand for $\Sigma_0 = S_0 - 2\hat{S}$. Note that from (7.6) and (3.5), these diagrams also sum to $\Lambda\partial S^1_{\mu\nu}(p)/\partial\Lambda - 2\beta_1\hat{S}_{\mu\nu}(p)$, and thus after Λ integration, yield the one-loop **AA** vertex.

Translating as described in section 4, since there is no tree-level vertex with a single **C**, we see that the wines in figure 13 attach via two **A**s, two **B**s or two **C**s. Expanding these attachments as in figure 2, the terms that survive have a σ_3 pair at the top or bottom of the wine (cf. the discussion below (2.12) — the terms with

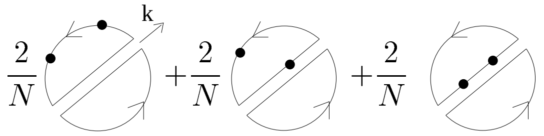


Figure 13: The one-loop two-point diagrams.

either both σ_3 pairs, or neither σ_3 pair, vanish by the supertrace mechanism). Thus we obtain:[11]

$$
\begin{aligned}
& a_1[\Sigma_0]^\sigma_{\mu\nu}(p) = \\
&= \frac{2}{\Lambda^2}\int \frac{d^Dk}{(2\pi)^D}\Big\{ c'_{k,\alpha\beta}\Sigma^0_{\alpha\beta\mu\nu}(-k,k,p,-p) + c'_{\mu,\alpha\beta}(p;k-p,-k)\Sigma^0_{\alpha\beta\nu}(p-k,k,-p) + \\
&\qquad + c'_{\mu\nu,\alpha\beta}(p,-p;k,-k)\Sigma^0_{\alpha\beta}(k) + \\
&\qquad + \frac{1}{\Lambda^2} M_k \Sigma^{0\mathbf{CCAA}}_{\mu\nu}(-k,k,p,-p)(p-k,k,-p) + \\
&\qquad + \frac{1}{\Lambda^2} M_\mu(p;k-p,-k)\Sigma^{0\mathbf{CCA}}_{\nu} + \frac{1}{\Lambda^2} M_{\mu\nu}(p,-p;k,-k)\Sigma^{0\mathbf{CC}}(k) - \\
&\qquad - K_{k,\alpha\beta}\Sigma^{0\mathbf{BBAA}}_{\alpha\beta\mu\nu}(-k,k,p,-p) - \\
&\qquad - K_{\mu,\alpha\beta}(p;k-p,-k)\Sigma^{0\mathbf{BBA}}_{\alpha\beta\nu}(p-k,k,-p) - \\
&\qquad - K_{\mu\nu,\alpha\beta}(p,-p;k,-k)\Sigma^{0\mathbf{BB}}_{\alpha\beta}(k)\Big\}\,,
\end{aligned}
\tag{8.3}
$$

where we have noted that, by Lorentz invariance, the $p_\mu \leftrightarrow -p_\nu$ transposition just yields a factor 2 [1].

The momentum integral is finite as required, for appropriately chosen covariantization and cutoff functions. We use (2.21) (evaluated in section 5) and keep c and $\tilde{c}$ general except for the requirements $c(0) = 1$ and $\tilde{c}(0) > 0$ discussed in section 2, and

$$
c(x) \propto x^{-r} \qquad \text{and} \qquad \tilde{c}(x) \propto x^{-\tilde{r}}\,, \tag{8.4}
$$

for large x. Physically, we are interested in $D = 4$ where g is marginal, and β_1 is universal. We set

$$
r > \tilde{r} > 1\,. \tag{8.5}
$$

This is helpful for calculation, but more stringent than necessary, since we show in the next section that physical one-loop contributions are finite for $r > \tilde{r} > 0$.

Note that although (8.3) is thus finite, it is the sum of integrals with cancelling divergences and therefore still ambiguous, a generic problem with Pauli-Villars regularisation [10]. We will keep D general and take the limit $D \to 4$ only at the end of the calculation. This amounts to dimensional preregularisation [10], and allows us to discard those parts which are surface terms in any dimension D, but keep those parts which become surface terms only in $D = 4$ dimensions; the independence of β_1 on the choice of cutoff functions, i.e. its universality, arises from being expressed *entirely* through such latter terms. For an explicit demonstration of these subtleties, see the example in ref. [3]. Further comments can be found later in this section and in the conclusions.

[11] Note that the wine vertices appear reflected in figure 13 compared to section 4.

By (7.6), in the two-point vertices of (8.3) we have $\Sigma_0 \equiv -\hat{S}$. Applying (6.2), the first term of (8.3) becomes

$$c'_{k,\alpha\beta}\Sigma^0_{\alpha\beta\mu\nu}(-k,k,p,-p) \equiv c'_k \Sigma^0_{\alpha\alpha\mu\nu}(-k,k,p,-p) + L_k \left\{ \hat{S}_{\mu\nu}(p) - \hat{S}_{\mu\nu}(k+p) \right\},$$

where we use the above comment, and the fact that $L_k k_\alpha$ vanishes by Lorentz invariance of k integral. The expression for $\Sigma^0_{\alpha\alpha\mu\nu}(-k,k,p,-p)$ derived from the last section, may be further simplified by using Lorentz invariance (in fact $k \leftrightarrow -k$, $\mu \leftrightarrow \nu$ etc.) and the coincident line identities (5.5) in a manner already described in ref. [1]. After these manipulations, we simply substituted all the expressions derived earlier, and expanded to $O(p^2)$ to extract β_1 via (8.2). The large number of resulting terms were handled with care by algebraic computing. (We used FORM.) This calculation is described below.

It seems likely that we could have proceeded more intelligently: after all, β_1 should be universal not only to choices of c and $\tilde{c}$ but also to the choices of wine vertex and seed action $\hat{S}$, i.e. covariantization (which, recall, need not even be the same in each wine). Therefore the result should fall out without the need for all these expressions to be explicit. We leave such investigations for the future.

A formula for the gauge dependent part falls out simply as follows, and acts as a check on the calculation [1]. Using (6.2) and (6.3), collecting terms using Lorentz invariance under $k \leftrightarrow -k$, and again using (7.6),

$$\begin{aligned} p^\mu p^\nu a_1[\Sigma_0]^\sigma_{\mu\nu}(p) = \frac{2}{\Lambda^2}\int \frac{d^Dk}{(2\pi)^D} \Bigg\{ & c'_{k,\alpha\beta}\hat{S}_{\alpha\beta}(k) + \frac{1}{\Lambda^2}M_k \hat{S}^{\mathbf{CC}}(k) - K_{k,\alpha\beta}\hat{S}^{\mathbf{BB}}_{\alpha\,\beta}(k) - \\ & - c'_{k+p,\alpha\beta}\hat{S}_{\alpha\beta}(k{+}p) - \frac{1}{\Lambda^2}M_{k+p}\hat{S}^{\mathbf{CC}}(k{+}p) + \\ & + K_{k+p,\alpha\beta}\hat{S}^{\mathbf{BB}}_{\alpha\,\beta}(k{+}p) \Bigg\} . \end{aligned} \tag{8.6}$$

By a shifting momentum $k \mapsto k+p$ in the first three terms we see that the gauge dependent part is zero as it must be. However, expanding (8.6) to order p^4 using (4.12), and as in the ensuing calculation, leaving the radial part of the k integral to last, we obtain

$$\begin{aligned} a_1[\Sigma_0]^\sigma_{\mu\nu}(p) = & -4(D-1)\Omega_D \Lambda^{D-4} \left\{ \Lambda^2 \delta_{\mu\nu} \int_0^\infty dx \frac{d}{dx} G_0 + p_\mu p_\nu \int_0^\infty dx \frac{d}{dx} G_L \right\} - \\ & - 4\beta_1 \Delta_{\mu\nu}(p) + O(p^3) \end{aligned} \tag{8.7}$$

where

$$G_0 = \frac{x^{D/2}F'}{D(D-1)}, \qquad G_L = \frac{1}{D(D+2)(D-1)} \left(x^{D/2+1}F'' \right)' \tag{8.8}$$

and

$$F = (D-1)(c'-K)\frac{x}{c} - \frac{(DK+xL)}{\tilde{c}} + \left(\frac{x}{\tilde{c}} + \sigma \right) M \,. \tag{8.9}$$

Here $\Omega_D = 2/[\Gamma(D/2)(4\pi)^{D/2}]$ is the solid angle of a $(D-1)$-sphere divided by $(2\pi)^D$, and as before prime is differentiation with respect to its argument (here x). The transverse part follows from (8.2). One readily verifies, cf. (9.5), that with $r > \tilde{r} > 0$, $F \sim 1/x^f$ as $x \to \infty$ with $f > 1$, and thus the $\delta_{\mu\nu}$ and $p_\mu p_\nu$ contributions to (8.9) do indeed vanish.

We sketch the main steps of the remaining calculation. After simplifying (8.3) as discussed earlier, and substituting for all S^0 and $\hat{S}$ four-point vertices, we simplify the special-momenta terms by recognizing that both $S^{0\,\mathbf{XXA}}_{a_1a_2\lambda}(q,-q,0)$ and $\hat{S}^{\mathbf{XXA}}_{a_1a_2\lambda}(q,-q,0)$ collapse to $\frac{\partial}{\partial q^\lambda}\hat{S}^{\mathbf{XX}}_{a_1a_2}(q)$, as follows from gauge invariance.[12] After substituting for all remaining vertices, the wine expressions (5.9), (5.11) and (5.12) are substituted, together with the special-momentum case (5.14). No other special-momenta terms requiring careful limits, arise.

We collected terms under the substitution $k \leftrightarrow -k$, and expanded to overall order p^2, which is now straightforward. The angular part of the k integration is achieved through the following equivalences under the integral

$$(k.p)^2 k_\mu k_\nu \equiv \frac{k^4}{D(D+2)}\left(p^2\delta_{\mu\nu} + 2p_\mu p_\nu\right) \qquad \text{and} \qquad k_\mu k_\nu \equiv \frac{k^2}{D}\delta_{\mu\nu}\,,$$

as follows from considering Lorentz invariance (of in the first case $k_\alpha k_\beta k_\mu k_\nu$). The result is written as a linear combination of $\delta_{\mu\nu}$, $p_\mu p_\nu$ and $\Delta_{\mu\nu}(p)$.

As well as the radial k integral, there are a number of Λ integrals to evaluate. The most involved of these are the integrals for the classical four-point vertices since these contain in turn, some inner Λ integrals for classical three-point vertices. Starting with these innermost Λ integrals, we substitute for K, L and M using (2.26), and expand the derivatives until they are all expressed directly in terms of differentiated cutoff functions, i.e. $c^{(n)}(x)$ and $\tilde{c}^{(m)}(x)$; note that these always appear in the numerator of the integrands.

The resulting expression is very large, yet it must boil down to the one number β_1. The game then is to manipulate it in such a way as to successively cancel out nearly all the terms. To do this, after any differentiation or integration we express the resulting terms in a unique algebraic form: in this way any algebraic cancellations take place automatically. Denominators involve positive integer powers of $b(x) = 1/(x\tilde{c}+c)$ and $f(x) = 1/(x+\sigma c)$ ($x = k^2/\Lambda^2$ being effectively the integration variable). Iteratively, every appearance of xb is replaced by $(1-cb)/\tilde{c}$, and every appearance of b/x by $(1/x - \tilde{c}b)/c$. These relations thus eliminate x or/and b in $x^m b^n$, in favour of c and $\tilde{c}$. A number of other ways of bringing such terms to a canonical form also exist, but this one has the advantage that it does not exchange terms which behave well as $x \to 0$ or ∞, with terms that individually do not. Similarly we tidy f terms via iterating $xf = 1 - \sigma\tilde{c}f$ and $f/x = (1/x - f)/(\sigma\tilde{c})$. Finally terms with both b

[12]Cf. the notation of (4.1); use (6.2) on $s^\lambda S_{\mu\nu\lambda}(q,r,s)$ and expand for small s.

and f in but no explicit power of x, are brought to canonical form by iterating the substitutions $\tilde{c}^2 bf = (cbf + \tilde{c}b - f)/\sigma$ and $cbf/\tilde{c} = \sigma\tilde{c}bf + f/\tilde{c} - b$.

We now integrate by parts the inner integrals as follows. We take first the terms with a $c^{(n)}(x)$ or $\tilde{c}^{(n)}(x)$ where the derivative n is the largest that appears. (For the inner integrals the maximum derivative is $n = 4$; overall, it is $n = 5$.) We then integrate by parts to systematically reduce this maximum n. If such $\tilde{c}^{(n)}$ terms also contain some positive power m of $\tilde{c}^{(n-1)}$, we eliminate $\tilde{c}^{(n)}$ by first writing

$$\tilde{c}^{(n)} \left(\tilde{c}^{(n-1)}\right)^m = \frac{1}{m+1}\frac{d}{dx}\left(\tilde{c}^{(n-1)}\right)^{m+1} \tag{8.10}$$

and similarly for c. If a $c^{(n-1)}$ multiplies the terms in (8.10), we still integrate by parts, thereby exchanging a $\tilde{c}^{(n)}$ for a $c^{(n)}$ — which we take to be the canonical order. Obstructions to further reduction of the maximum derivative n thus occur if there are terms with a product of more than one n-derivative factor (i.e. $c^{(n)}$ or $\tilde{c}^{(n)}$) or if the product $c^{(n)}\tilde{c}^{(n-1)}$ appears. Although such terms have the potential to exist they all cancel away as we iterate n down to $n = 1$.

Incidentally, as mentioned in section 7, we also derived β_1 by keeping an explicit initial Λ_0. Integrating by parts then often yields negative powers of Λ_0 from the $\Lambda = \Lambda_0$ boundary (if necessary by Taylor expansion in $1/\Lambda_0$). At this stage it is straightforward to estimate the maximum divergence of factors containing further Λ integrals, as a power of Λ_0 (or as $\ln\Lambda_0$), and thus we drop all terms that go overall as a negative power of Λ_0; in fact in this way we find that at most a single power of $1/\Lambda_0^2$ need be kept.

When there are only first derivatives left the strategy for a unique simplification route is more involved because differentiating b (f) produces $\tilde{c}'$ and c' ($\tilde{c}'$). We solved this with the following algorithm. If the integrand contains a single c', and otherwise only c and x then the final derivative can be eliminated similarly to (8.10) (introducing $\ln c$ if $m = -1$). A similar strategy applies for $\tilde{c}'$, except that we allow a power of c — thus exchanging $\tilde{c}'$ terms for those with c', our canonical choice. Next, if there is no c', one $\tilde{c}'$ and a factor of b^m where $m \geq 2$ (there are such terms up to $m = 5$) then we write:

$$b^m\tilde{c}' = \frac{1}{m-1}\frac{\tilde{c}}{c}\frac{d}{dx}b^{m-1} + b^m\frac{(\tilde{c}+c')}{c} + b^{m-1}\frac{\tilde{c}'}{c} \qquad (m \geq 2)\,,$$

integrate by parts the explicit d/dx, and iterate until there are no more such terms. Next, if there is no c', no b, one $\tilde{c}'$ and a factor of f^m where $m \geq 2$ we write

$$f^m\tilde{c}' = -\frac{1}{\sigma}\left(f^m + \frac{1}{m-1}\frac{d}{dx}f^{m-1}\right) \qquad (m \geq 2)\,,$$

integrate by parts the explicit d/dx, and iterate until there are no more such terms. After all this is done (mindful always to map the results to a unique algebraic form as above) no further simplification is possible on these integrals.

For the $\delta_{\mu\nu}$ and $p_\mu p_\nu$ terms, these manipulations are sufficient to evaluate all the inner Λ integrals and then the outer integrals; we checked that the remaining integral, the radial k integral, indeed takes the form (8.9) as expected. (In fact we also checked that this is so, independent of (2.26), again as expected.)

For the $\Delta_{\mu\nu}(p)$ terms however, there are quite a few inner Λ integrals with at most singly differentiated cutoff functions, that cannot be further simplified. At this stage we integrate by parts the outer Λ integral (and single Λ integrals), and simplify as much as possible, proceeding in a similar manner to above. (We need also to eliminate c' in $c' \ln c = (c \ln c - c)'$ etc.) This causes some of the remaining inner Λ integrals to disappear by differentiation.

Finally, we convert the measure in the k integral to

$$\Lambda^D \Omega_D \int_0^\infty dx\, x^{D/2-1}\,, \tag{8.11}$$

where $x = k^2/\Lambda^2$, and integrate this by parts, again in the manner sketched above.

In the case that we simply set $D = 4$ from the beginning we find that *all* the remaining integrals are thus evaluated, converting the expression to a set of boundary terms. In this way, the remaining calculation consists of a set of limits: $x \to 0$ and $x \to \infty$ from the boundaries of the radial k integral (and first $x_0 := k^2/\Lambda_0^2 \to 0$ if the explicit initial Λ_0 was kept). These limits may themselves involve integrals, however at this stage these integrals are easily done by substituting the appropriate asymptotic behaviour for the integrands as $x \to 0$ or ∞. Indeed, just by power-counting, the vast majority of the remaining terms simply vanish. Let us give just one example. In order to be finite under (8.5), at one stage we have to collect together some separately divergent terms. Together these yield, using (8.2), a contribution to β_1 of

$$\frac{1}{(4\pi)^2} \lim_{y\to\infty} \left\{ 8yc(y)\tilde{c}(y) \left(\int^y dx \frac{[1/x - b(x)\tilde{c}(x)]}{c(x)} \right)^2 - \right.$$
$$\left. - [15y\tilde{c}(y) + 3y^2\tilde{c}'(y)] \int^y dx \frac{[1/x - b(x)\tilde{c}(x)]}{c(x)} \right\}. \tag{8.12}$$

In here the lower limit of the integrals corresponds to $\Lambda = \Lambda_0$ and is evaluated first. It is either explicitly finite by cancellation from other parts or the divergences are discarded in the manner already described in section 7. Under the ultraviolet limit $y \to \infty$, the first term vanishes and the second yields

$$\frac{1}{(4\pi)^2} \left(\frac{3}{\tilde{r} - 1} - 12 \right),$$

as follows from (8.4).

After all the limits are evaluated in the $D = 4$ calculation, we obtain an expression for β_1 which is independent of all the extra parameters γ introduced in the previous section,[13] and all the details of c and $\tilde{c}$ except that there is still a dependence on the powers r and $\tilde{r}$:

$$\beta_1 = \frac{1}{(4\pi)^2}\left(16\frac{\tilde{r}}{r}+\frac{8}{r}+\frac{1}{1-\tilde{r}}-r^2-\tilde{r}^2+r\tilde{r}-r+\frac{\tilde{r}}{2}-\frac{205}{6}\right).$$

Nevertheless the dependence on r and $\tilde{r}$ means that it is therefore not universal — a clear signal that we can attach no physical significance to the result. Indeed, we have already seen [3] that the result will alter if different momentum routings are chosen (e.g. shifting k by $\pm p$) in different parts of (8.3).

We return to step (8.11), and leave D general, setting $D = 4-\epsilon$. Integrating by parts and expanding in small ϵ, we now obtain some extra remainder terms which are ϵ times integrals that diverge as $D \to 4$. (As in dimensional regularisation terms with any higher power of ϵ are easily seen to vanish at one loop as $\epsilon \to 0$.) We do not expect it to make any difference whether ϵ is taken positive or negative since the integral is finite for any small ϵ providing we are allowed to shift momenta and discard the resulting surface terms. Rigorously however, we choose $D < 4$ since then all surface terms vanish for D small enough, after which the result may defined by analytic continuation to all D. In the present case any value of $D < 4$ is sufficient. Throwing away those terms that clearly vanish for $D < 4$, and restoring the factor N absorbed by the large-N change of variables [1], we find [3]:

$$\begin{aligned}\beta_1 \;=\; & \frac{N}{(4\pi)^2}\frac{\epsilon}{2}\Bigg[\int_0^\infty dx x^{\epsilon/2-1}\{2+4c^2-6c\}+\int_0^\infty dx x^{\epsilon/2}\left\{\frac{59}{2}\frac{\tilde{c}}{x\tilde{c}+c}-\frac{59}{6}\frac{1}{x+\sigma\tilde{c}}\right\}+\\ & +\int_0^\infty dx\; x^{\epsilon/2}\,\tilde{c}\int_0^x dx_1\left\{\frac{20\tilde{c}}{(x_1\tilde{c}+c)^2}-\frac{2}{\tilde{c}(x_1+\sigma\tilde{c})^2}+\frac{24\tilde{c}'}{\tilde{c}(x_1\tilde{c}+c)}-\frac{6\tilde{c}'}{\tilde{c}^2(x_1+\sigma\tilde{c})}\right\}\Bigg]\\ \to\; & -\frac{11}{3}\frac{N}{(4\pi)^2} \qquad \text{as} \qquad \epsilon \to 0\,, \end{aligned} \tag{8.13}$$

the famous one-loop result for SU(N) Yang-Mills, completely independent of the γs and c and $\tilde{c}$, this time including r and $\tilde{r}$.

(N.B. We display in (8.13), as an example, the penultimate terms in a way closely similar to those that we computed. There is no sense however of a canonical choice here: they are easily altered by adding terms whose differences obviously vanish as $D \to 4^-$. Only the final limit is invariant.)

9. One loop finiteness

For the following proof, it is helpful to constrain the form of the covariantization and to specify the general form of the asymptotic behaviour of the cutoff functions. We specialize to covariantizations that are minimal (in the sense of ref. [1, appendix A]), such as but not exclusively (2.21), and to cutoff functions where the scale is set by

[13] Ref. [3] illustrates this by showing how the $\left(\gamma^{BBA}\right)^2$ term disappears.

the momentum if it is much greater than Λ which implies here that they decay as a power for large momenta (cf. again [1, appendix A]), i.e. are of form (8.4) for large x.[14] The conditions that c and $\tilde{c}$ decay, and (2.25), imply

$$r > 0\,, \qquad \tilde{r} > 0 \qquad \text{and} \qquad r > \tilde{r} - 1\,. \tag{9.1}$$

In this section we prove the following theorem, which relies also on the more stringent conditions:

$$\Delta r > \frac{D-2}{2} \tag{9.2}$$

where we set $\Delta r := r - \tilde{r}$, and

$$\tilde{r} > \frac{D-2}{2}\,. \tag{9.3}$$

Theorem 1. *Conditions (9.1), (9.2) and (9.3) are necessary and sufficient (at $N = \infty$) to ensure that the one-loop corrections in the flow equation (2.27), with any number of external* **A** *fields and no external* **B** *or* **C** *fields, are finite.*

Note that by replacing D in (9.2) and (9.3) by $D + \Delta$ we can thus obtain for the one-loop momentum integral over k, any degree of convergence $\sim k^{-\Delta}$ that we desire. In fact, as we discuss later, for those one-loop corrections other than the vacuum amplitude (i.e. for those corrections containing at least one external **A**), we can relax (9.2) and (9.3) to

$$\Delta r > \frac{D-4}{2} \qquad \text{and} \qquad \tilde{r} > \frac{D-4}{2}\,. \tag{9.4}$$

In particular in four dimensions, $r > \tilde{r} > 0$ is sufficient for our purposes.

Clearly the cases considered are sufficient for the computation of physics to one loop. For compactness we will refer to contributions without external **B**s or **C**s as *BC bereft.* Note that the supertraces will generically still contain trapped σ_3s as well as external **A** fields. It may be possible to lift the restriction on no external **C**s, which is made here for convenience. On the other hand we show at the end of this section that in the present formulation, there are one-loop corrections with external **B**s that diverge, and there are divergent diagrams beyond one loop.

Before launching into the proof proper, we sketch the main reasons for finiteness. This in turn helps to explain why the flow equation (2.27) takes the form it does. See also ref. [3]. Again, it is helpful to refer to the three terms in (2.23) as $\hat{S} = \hat{S}_A + \hat{S}_B + \hat{S}_C$, respectively. It is also helpful to refer to the different terms in (2.27) by their covariantized kernels, i.e. as the attachment of the associated wines $\{c'\}$, $\{L\}$, $\{K-c'\}_{\mathbf{A}}$ and $\{M-L\}$. (Thus here we will not use the fact that the covariantization is linear e.g. $\{M-L\} = \{M\} - \{L\}$ which implies the **C**-**C** attachment is actually via $\{M\}$).

Apart from ensuring that the high energy behaviour of the Feynman rules is as expected by the SU($N|N$) properties (this is discussed in the next three paragraphs below), there are three main mechanisms for finiteness which are essentially

[14] I.e. in the sense that $x^r c(x)$ and $x^{\tilde{r}} \tilde{c}(x)$ attain finite non-zero limits as $x \to \infty$.

very simple: One is the supertrace mechanism that we have already discussed in section 2. Another is 'drifting' [3]: many potentially distastrously divergent contributions following from attaching L in (2.27) via the $\mathcal{A}$ differential in l on or near $\hat{S}_A$, disappear by gauge invariance. These two mechanisms are sufficient to ameliorate the problems [8] caused by covariantizing the higher derivatives. The final mechanism for finiteness is simply the existence of the higher derivatives themselves. As we will see the proof of finiteness is nevertheless sufficiently involved to make a more mathematical style convenient. We expect that much simpler proofs will be possible in a manifestly local SU($N|N$) framework.

We choose the kernels (2.26) so that the kinetic terms (bilinears in the fields) in $\hat{S}$ and S can coincide as in (7.6), and thus lead to the simplifications we saw in section 7, mimicking ref. [1]. This is all the more desirable in the PV sector because we want the Fermi-Bose cancellations that regularise diagrams with $\hat{S}_{\mu\nu}$ in, to work also on those diagrams with $S_{\mu\nu}$. In fact the problem would be worse than this because when the two-point functions do not coincide, and thus the cancellations [1] described in section 7.2 do not take place, the higher point functions of S pick up integration factors that can radically alter their ultraviolet properties. With the cancellations described in section 7.2, diagrammatic contributions may be immediately integrated with respect to Λ, which means in particular that their ultraviolet properties follow more or less straightforwardly from the component vertices, just as they do in the usual application of Feynman rules. (Their leading ultraviolet properties are the same as those of the integrand providing that the coefficient of the leading ultraviolet behaviour of the integrand also converges when integrated over Λ.)

At the same time however, we need the kernels themselves to have the right normalisations to lead to cancellations at high momentum just as the propagators do (as in ref. [3], or via the spontaneously broken SU($N|N$) interpretation). Since the kernels are essentially the Λ differentials of these propagators, we need care in choosing the powers of Λ multiplying these propagators, equivalently the powers of Λ in front of the $\hat{S}$ two-point functions in (2.23), or again equivalently, the natural dimensions of the fields. This requires the assignment of mass dimension one for **B** and the less conventional choice of dimension zero for **C**.

With these choices we have as required that the transverse **B**'s kernel K and **A**'s kernel c' coincide at high energy, and the longitudinal **B**'s kernel L and **C**'s kernel M coincide at high energy. Indeed from (9.1) and (2.26),

$$\begin{aligned} K(x) &= c'(x)\left\{1+O\left(\frac{1}{x^{1+\Delta r}}\right)\right\} \\ L(x) &= \frac{(x\tilde{c})'}{x}\left\{1+O\left(\frac{1}{x^{1+\Delta r}}\right)\right\} \\ M(x) &= \frac{(x\tilde{c})'}{x}\left\{1+O\left(\frac{1}{x^{1+\tilde{r}}}\right)\right\}. \end{aligned} \tag{9.5}$$

This high momentum restoration of the unbroken SU($N|N$) clearly improves the ultraviolet behaviour of the last two terms in (2.27). Meanwhile, quantum corrections following from differentiating the full $\mathcal{A}$, will cancel by the supertrace mechanism (2.8), unless a σ_3 is trapped in between. A slightly more subtle supertrace mechanism works between **B** and **C** differentials of $\mathcal{B}$ (reflecting the fact that in essence, **B** ate **D**).

As we have already mentioned, this supertrace mechanism together with the drifting property cure the problems associated with covariantizing the higher derivatives. Indeed, as we now show, the conditions (9.2) and (9.3) are anyway required to ensure that there are sufficient higher derivatives to regularise, independently of the problems caused by covariantization. (Note (9.1), which is also required, has already been discussed in section 2.)

Lemma 2. *Conditions (9.2) and (9.3) are necessary for ultraviolet finiteness.*

Proof. Consider the vacuum contribution to the simple one-loop vacuum diagram $a_1[\hat{S}_C]$, as illustrated in figure 14, where the wine is L. Clearly there is no supertrace cancellation since there is no analogous **B** term in (2.23). By (8.4) and (9.5), this contribution is UV finite if and only if (9.3) holds.

Similarly, consider $a_1[\hat{S}_B]$ where the kernel is c' which we attach to the lone **B** in the $\mathcal{B}$s. Again there is no supertrace cancellation, because there is no corresponding longitudinal **A** term in (2.23). By (8.4), this contribution is UV finite if and only if (9.2) holds. $\square$

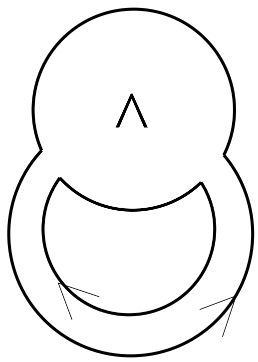


Figure 14: Simple one loop diagram.

Note that (9.2) and (9.3) are strictly necessary in the above examples only for the vacuum contributions. Discarding vacuum energy contributions, the first non-trivial contribution in figure 14 must be the two-point contribution since the one-point contribution vanishes by $\mathrm{tr}\, A^i = 0$ (as well as many other reasons [1]). By sections 4 and 5.5 (or more generally ref. [1, appendix A]), wherever the two points are placed, the large momentum behaviour is improved by a power of two. Therefore in (9.2) and (9.3) we can replace D by $D-2$, obtaining (9.4). By keeping careful track of where the external points must be, we can prove these more relaxed conditions for the general contributions, but we will not pursue it further here: the important point we wish to establish is that conditions *do* exist that guarantee finiteness for all physically relevant one-loop graphs.

We have already shown that there are no terms of the form of figure 3. Therefore a one loop graph is formed only by attaching wines to *other* wines, or lobes $\hat{S}$, as shown in figure 15.

Here we have used the fact that one of the two superloops must be field free to survive the large-N limit. Without loss of generality, we take it to be the inner

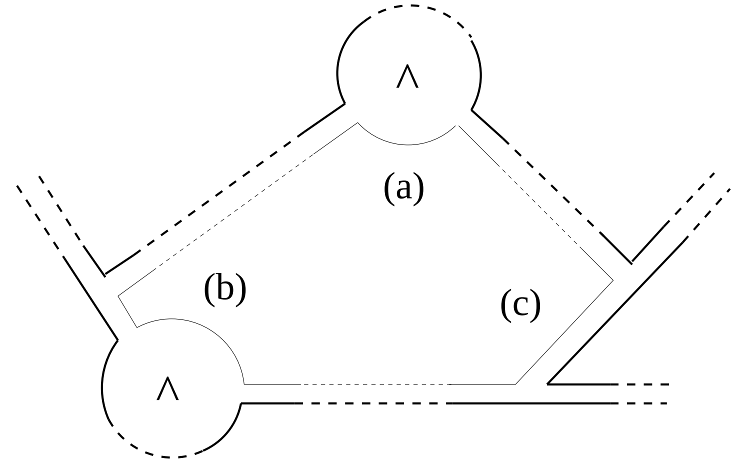


Figure 15: General form of the one loop diagram.

Wilson loop. It follows that the inner loop also grows no tree corrections, because tree corrections have at least one lobe with only one wine attached, and these lobes must have at least one external field because there are no $\hat{S}$ one-point vertices.

As illustrated in figure 15, the one loop contribution will have one or more of the following features:

(a) The loop momentum k is carried directly into and out of a lobe by wines. The lobe $\hat{S}$ can have any number of external fields on the outer Wilson loop and any number of tree corrections branching off it.

(b) As in (a) except that one or both wines carrying k, first attach to another wine, the rest of which is attached to some tree correction.

(c) The loop momentum passes through three wines (in one of several ways) with no intermediate lobe.

At first sight it is easy to construct a divergent one-loop diagram: we use the $\hat{S}_A$ terms for the lobes and $\{L\}$ for the wines. By (9.1) or lemma 2, the $k^{-2\tilde{r}}$ is in general no way sufficient to soften the divergent k^{2r} behaviour in $\hat{S}_A$. However, the L attachment must be made via the $\nabla_\mu \cdot \delta/\delta\mathcal{A}_\mu$ part of l, there being no $\mathbf{C}$ terms in $\hat{S}_A$ or the wines (where attachment may be made as in figure 15b). Gauge invariance then implies drifting which as we will see, entirely cancels all such dangerous contributions (in BC bereft one-loop diagrams).

Thus, consider an $\{L\}$ wine attaching to a pure $\mathcal{A}$ section of a superloop. The attachment takes the form

$$\int d^D w \,\mathrm{str}\left[\mathbf{X}(w)\nabla_\mu \cdot \frac{\delta}{\delta\mathcal{A}_\mu(w)}\right]\Phi_{xy} = -\int d^D w\,\mathrm{str}\left[\nabla_\mu \cdot \mathbf{X}(w)\frac{\delta}{\delta\mathcal{A}_\mu(w)}\right]\Phi_{xy}$$
$$= i\left\{\Phi_{xy}\mathbf{X}(y) - \mathbf{X}(x)\Phi_{xy}\right\}, \qquad (9.6)$$

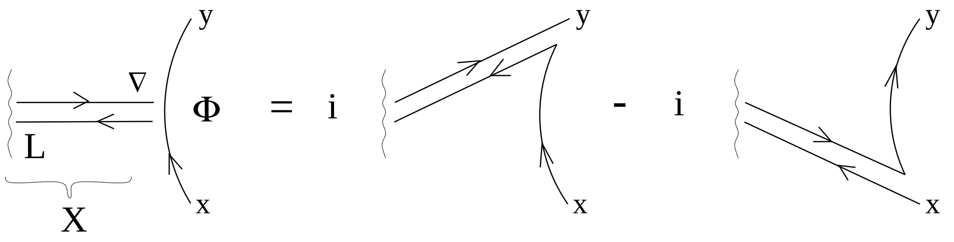


Figure 16: Drifting or sliding $\{L\}$.

where $\mathbf{X}(w)$ contains $\{L\}$ and all that it attaches to at its other end, $\Phi_{xy}[\mathcal{A}]$ is the pure gauge section, with ends at x and y, and the last line follows since Φ_{xy} transforms homogeneously under gauge transformations (2.3). The interpretation in terms of drifting, or sliding, is now clear from figure 16.

The obstructions at the ends of Φ_{xy} are σ_3s, $\mathbf{C}$s, explicit $\mathbf{B}$s,[15] or 'remaindered' $\mathbf{B}$s which arise in the following sense. The pure gauge section may depend only on $\mathbf{A}$ rather than the full supermatrix $\mathcal{A}$, as in $\{K - c'\}_{\mathbf{A}}$ of (2.27). Drifting still occurs by the first gauge transformation relation (2.3), but because $\{L\}$ attaches with the covariant derivative $\nabla = \mathbf{D} - i\mathbf{B}$, we obtain remainder terms with a $\mathbf{B}$ at the join. Equivalently we can write the pure $\mathbf{A}$ section as $\Phi_{xy}[\mathcal{A} - \mathbf{B}]$, so that Φ_{xy} expands into a $\Phi_{xy}[\mathcal{A}]$ plus a series of smaller pure $\mathcal{A}$ sections with remainder $\mathbf{B}$s at one or both ends.

If we replace the section $\Phi_{xy}[\mathcal{A}]$ in (9.6) by a full pure gauge Wilson loop $\varphi[\mathcal{A}]$,[16] then since this is gauge invariant, we find that attaching $\{L\}$ just results in zero. This is equally clear from figure 16 by sowing the two ends of Φ_{xy} together, i.e. by setting $x = y$ and taking a supertrace.

$\{L\}$ can drift of course across a join between two sections, say α and β, in a composite Wilson loop (e.g. drifting off a wine and onto a lobe). We can see this either by treating the two sections together as one pure gauge section Φ in (9.6), or more carefully as follows. We can add together the diagrams with $\{L\}$ attached to α, and with $\{L\}$ attached to β. By (9.6) and figure 16, the drift to the join from α cancels the drift to the join from β.

Note however: we are making an assumption when sliding $\{L\}$ across a join, that the new diagram actually exists, i.e. can be constructed from (2.27). In figure 17 we show an exception.

We take the diagram to have no external $\mathbf{B}$s. The lobe is $\hat{S}_B$ and the $\{c'\}$ wine attaches by differentiating the lone $\mathbf{B}_\mu$ in $\mathcal{B}_\mu$. The $\{L\}$ attaches to $\{c'\}$, absorbing the $\mathbf{B}$ that has to be there by fermion number (6.7), and then slides back to the join. The corresponding diagram with $\{L\}$ attached to $\hat{S}_B$, but all other factors the

[15] Which may be regarded as arising from σ_3s by writing $\mathbf{B} = \mathrm{d}_- \mathcal{A}$, cf. (2.9).

[16] Recall that we can think of these pure gauge contributions as fluctuating Wilson loops, integrated over with some suitable measure [1].

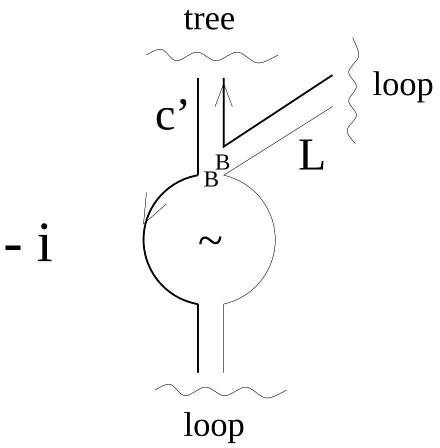


Figure 17: A diagram with no external $\mathbf{B}$s where $\{L\}$ cannot slide across the join.

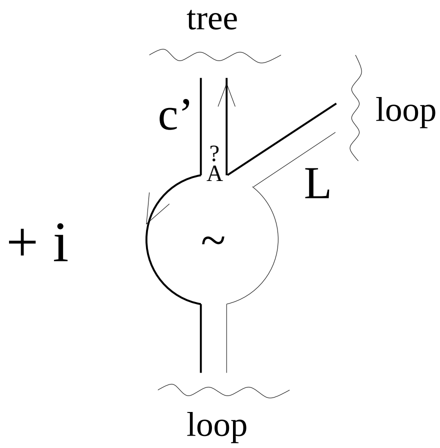


Figure 18: A diagram with no external $\mathbf{B}$s and drifted $\{L\}$, that does not exist.

same, does not exist however: since $\{c'\}$ and above is $\mathbf{B}$-free, in the corresponding diagram $\{c'\}$ must attach by differentiating an $\mathbf{A}_\mu$ in the same place as the lone $\mathbf{B}_\mu$ as illustrated in figure 18, but there is no such $\mathbf{A}_\mu$ term.

(Such a longitudinal $\mathbf{A}_\mu$ term is not allowed by gauge invariance.) Fortunately, as we see in this example, firstly the problem only arises where a bosonic attachment must change into a fermionic one (or vice versa) as a consequence of sliding a $\delta/\delta\mathbf{B}$ across the join, and secondly the problem is always flagged by the appearance of σ_3s at the join expressing the fact that the attachment is only via a partial supermatrix as in figure 2.

Collecting the observations above, we thus obtain the following lemma.

Lemma 3.

(a) $\{L\}$ *wines drift to the ends of pure* $\mathcal{A}$ *sections of the superloop.*

(b) $\{L\}$ *wines cannot attach to pure* $\mathcal{A}$ *Wilson loops.*

(c) $\{L\}$ *wines drift to the ends of pure* $\mathbf{A}$ *sections up to terms involving remaindered* $\mathbf{B}$*s.*

(d) $\{L\}$ *wines attach to pure* $\mathbf{A}$ *Wilson loops, in particular BC bereft tree-level contributions, only if accompanied by remaindered* $\mathbf{B}$*s.*

Proof. We only need note that in (d) a (BC bereft) tree-level contribution contains no σ_3s since they occur in pairs and thus may be commuted past $\mathbf{A}$s and combined together until they all disappear (cf. figure 2 and section 7), and $\{L\}$ has to attach by a bosonic differential since there are only $\mathbf{A}$s to differentiate, so diagrams exist for $\{L\}$ attaching at all points on the tree level contribution. $\square$

Lemma 4. *There are no tree-level contributions with just one* $\mathbf{B}$ *or one* $\mathbf{C}$*.*

Proof. We stress that this holds, whatever the number of $\mathbf{A}$s. That there are no one-point $\mathbf{B}$ diagrams is obvious, from the fermion number conservation (6.7) already mentioned. If an external $\mathbf{C}$ appears it must do so in $\hat{S}_B + \hat{S}_C$. But then together with any number of $\mathbf{A}$s, it accompanies in the $\hat{S}$ lobe, either two $\mathbf{B}$s which at tree level cannot be removed by further attachments (again by fermion number), or another $\mathbf{C}$. This other $\mathbf{C}$ can be propagated to other similar lobes by $\{L\}$ or $\{L-M\}$ in (2.27) where it may be turned into pairs of $\mathbf{B}$s (which again translate to external $\mathbf{B}$s), but it cannot disappear in any other way except apparently by attaching $\{L\}$ where we use the $\mathbf{C}$ differential in l, and at the other end we use the $\nabla_\mu \cdot \delta/\delta\mathcal{A}_\mu$ differential. However, this latter must attach to a BC bereft tree-level contribution and produce no remainder $\mathbf{B}$s, which is excluded by lemma 3(d). □

Corollary 5. *(a) BC bereft trees and BC bereft tree corrections are made only of lobes* $\hat{S}_A$ *and wines* $\{c'\}$*.*

(b) In a BC bereft one-loop diagram $\{K-c'\}_{\mathbf{A}}$*,* $\{M-L\}$ *and* $\{L\}$ *must appear entirely within the loop i.e. such that the loop momentum* k *goes in one end of the wine and out the other, while* $\hat{S}_B$ *(*$\hat{S}_C$*) must have the loop momentum enter and leave via the lone* $\mathbf{B}$ *or the* $\mathbf{C}$ *in* $\mathcal{B}$ *(*$\mathbf{C}$*).*

Proof. (a) For a BC bereft tree or BC bereft tree correction, if there existed $\hat{S}_B$ or $\hat{S}_C$, then breaking open the diagram by removing the wine connection to the $\mathcal{B}$ or $\mathbf{C}$ respectively, results in a tree with just one $\mathbf{B}$ or $\mathbf{C}$ in violation of lemma 4. Similarly if the BC bereft tree (correction) contained $\{K-c'\}_{\mathbf{A}}$, $\{M-L\}$ or $\{L\}$, then breaking open the diagram at the (tree) end of one of these wines we obtain a tree with one $\mathbf{B}$ and no $\mathbf{C}$s, one $\mathbf{C}$ and no $\mathbf{B}$s, and either one $\mathbf{C}$ and no $\mathbf{B}$s or neither $\mathbf{B}$s nor $\mathbf{C}$s, respectively. All these possibilities are excluded by lemmas 4 and 3(d).

(b) For the wines this is just a restatement of (a). For $\hat{S}_B$ and $\hat{S}_C$, if the loop momentum did not enter and leave in this way we would have to have an external $\mathbf{B}$ or $\mathbf{C}$ since by lemma 4, they cannot be absorbed by attaching a tree correction. □

Proposition 6. *In a BC bereft one-loop diagram,* $\{L\}$ *can only attach either:*

(a) directly to the lone $\mathbf{B}$ *or* $\mathbf{C}$ *in* $\mathcal{B}$ *of* $\hat{S}_B$*, or directly to the* $\mathbf{C}$ *in* $\hat{S}_C$*,*

(b) to $\{c'\}$ *as in figure 17, or*

(c) to $\{c'\}$ *or* $\hat{S}_A$ *as in figure 19.*

Proof. By corollary 5(b), if $\{L\}$ attaches to a lobe it either does so as in (a), or attaches to $\hat{S}_A$. However, it will drift off $\hat{S}_A$ unless it meets an obstruction which may be regarded as being caused σ_3 (cf. discussion above lemma 3). This in turn would arise only from a wine attached to $\hat{S}_A$ by a partial supermatrix derivative, uniquely specifying the attachment of $\{K - c'\}_{\mathbf{A}}$ and hence figure 19. Otherwise $\{L\}$ must attach to a wine, which by corollary 5(b) and figure 15, must be $\{c'\}$. If $\{L\}$ is not to drift off $\{c'\}$, it must meet an obstruction which can only be due to either a partial supermatrix derivative attachment, again giving uniquely $\{K - c'\}_{\mathbf{A}}$ as in figure 19, or $\{c'\}$ attaching to a partial supermatrix, which uniquely specifies figure 17. (By corollary 5, $\{c'\}$ cannot attach to $\{K - c'\}_{\mathbf{A}}$ since one end of the latter would be outside the momentum loop.) □

Note that (since $\{L\}$ is entirely inside the loop) if $\{L\}$ fails to find any obstruction then it will slide up its own tail as in figure 3, which we saw in section 2 gives zero. Equivalently, if it fails to find an obstruction then there are no trapped σ_3s, but since $\{L\}$ attaches by full supermatrix differentials, the result vanishes by the supertrace mechanism.

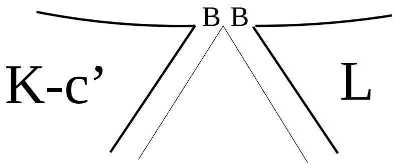


Figure 19: An $\{L\}$ attached to $\{c'\}$ or $\hat{S}_A$ which has drifted to touch a $\{K - c'\}_A$ wine.

In figure 19, by corollary 5(b), the loop momentum enters and leaves via $\{L\}$ and $\{K - c'\}_{\mathbf{A}}$. Since these two wines meet at the same point, the pure gauge vertex above (viz. $\hat{S}_A$ or $\{c'\}$) carries no loop momentum. Similarly in figure 17, by corollary 5(b), the loop momentum enters and leaves through $\{L\}$ and $\hat{S}_B$, and $\{c'\}$ carries no loop momentum. We see that drifting has entirely removed the dangerous combinations of $\{L\}$ and $\hat{S}_A$ discussed above (9.6), as promised.

By figure 15, there are at least as many wines carrying loop momentum as there are lobes carrying loop momentum. The insertion of extra wines as in figure 15b and figure 15c, cannot lessen the superficial[17] convergence. To be more precise, note that in the case where there is only one lobe carrying loop momentum (for example as in figure 14), there is always at least one 'internal' wine, in the sense of an internal propagator, i.e. a wine carrying loop momentum k from end to end. In a one-loop diagram with more than one k carrying lobe, there is always at least one such internal wine between any adjacent pair of such lobes. These internal wines are therefore at least equal in number to the lobes carrying k. The extra wines, over and above these internal wines, would be only partially within the loop, but by (5.20) and (5.24), for large momentum k they contribute at worst $\sim k^0$.

Consider a k-carrying lobe and all the wines on one side of the lobe that k passes through until the next k-carrying lobe is reached (or the same lobe is reached, if there is only one k-carrying lobe in the loop). Such a combination forms a dressed internal tadpole, which we will refer to simply as a tadpole. Note that such a tadpole thus always includes at least one internal wine. Every one-loop diagram can be split up (in one of two ways) into a set of disjoint tadpoles as illustrated in figure 20.

[17] I.e. as established by power counting and not taking into account possible cancellations.

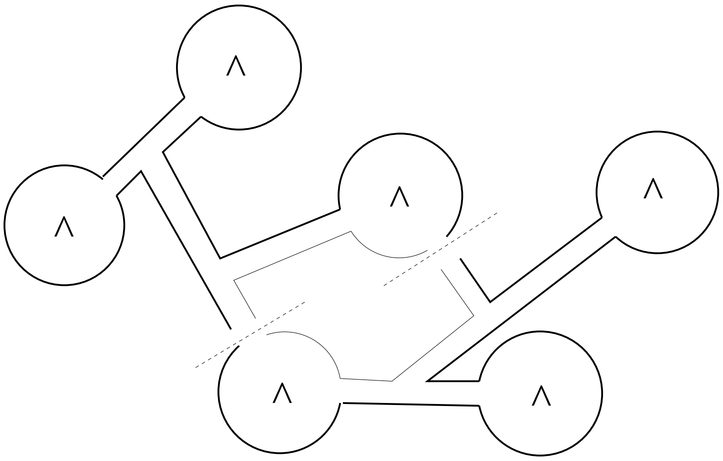


Figure 20: An example one-loop diagram split into two tadpoles by the dashed lines.

For a given tadpole, and large k, let k^{2p} be the power contributed by the k-carrying lobe, and let k^{-2w} be the total power of k contributed by the wines (again as established by power counting). We now use (8.4) and the results of section 5.5 (more generally of [1, appendix A]), to compute upper bounds for the tadpole's superficial degree of divergence $2(p-w)$, for each choice of lobe and choices of internal wine.

(i) Suppose that the k-carrying lobe is $\hat{S}_A$, then the maximum p that this can contribute is $p = r + 1$.

(i*a*) $\{M - L\}$ cannot be an internal wine in this tadpole since it joins directly to $\hat{S}_B + \hat{S}_C$ by (2.27) and corollary 5(*b*).

(i*b*) If $\{K - c'\}_{\mathbf{A}}$ appears as an internal wine then by (9.5),

$$p - w \leq -1 - \Delta r\,. \tag{9.7}$$

(i*c*) If $\{L\}$ is an internal wine in this tadpole then so is $\{K - c'\}_{\mathbf{A}}$ by proposition 6, figure 19 and corollary 5(*b*). (*N.B.* As noted above, the vertex shown above $\{K - c'\}_{\mathbf{A}}$ and $\{L\}$ in figure 19 *is not* carrying k.) Therefore (9.7) at least applies.

(i*d*) Finally $\{c'\}$ could be an internal wine, thus contributing a minimum of $w = r + 1$, and hence $p - w \leq 0$.

(ii) Now suppose that the k-carrying lobe is $\hat{S}_B$. This will contribute a maximum $p = \tilde{r}$ if attached to the loop by the **B**s, or $p = \tilde{r} + 1$ if attached to the loop via the **C**s.

(ii*a*) If $\{c'\}$ is an internal wine in this tadpole then by corollary 5(*b*) (and lemma 4), the $\hat{S}_B$ must attach via the **B**s and $p - w \leq -1 - \Delta r$.

(ii$b$) Similarly if $\{K-c'\}_{\mathbf{A}}$ is an internal wine, then by (9.5), $p-w \leq -2-2\Delta r$.

(ii$c$) If $\{M-L\}$ appears as an internal wine then this attaches to a $\mathbf{C}$ in $\hat{S}_B$, and by (9.5), $p-w \leq -1-\min(\tilde{r}, \Delta r)$.

(ii$d$) Finally if $\{L\}$ is an internal wine in this tadpole then either it can attach directly to the $\mathbf{C}$s in which case $p-w \leq 0$, or attach via the $\mathbf{B}$s in which case again $p-w \leq 0$.

(iii) Finally, suppose that the k-carrying lobe is $\hat{S}_{\mathbf{C}}$. This gives $p=0$. The least convergent internal wine is $\{L\}$, thus we have that whatever internal wines there are, $p-w \leq -1-\tilde{r}$.

Collecting these observations together we have

Proposition 7. *In a BC bereft one-loop diagram, all tadpoles have $p-w \leq 0$.*

Proof. The result follows trivially from the above observations on using the second two conditions of (9.1). □

Proposition 8. *Any BC bereft one-loop diagram containing a $\{K-c'\}_{\mathbf{A}}$ wine, or $\{M-L\}$ wine, or $\hat{S}_B$ and an internal $\{c'\}$, or $\hat{S}_C$, is finite.*

Proof. If $\{K-c'\}_{\mathbf{A}}$ appears in the diagram, it does so as an internal wine by corollary 5(b). Splitting the diagram into tadpoles, we have by (ib), (ii$b$) and (iii) above, and proposition 7, that the degree of divergence of the diagram is no worse than $D-2-2\Delta r$, and thus the diagram is finite by (9.2).

Similarly if $\{M-L\}$ appears, it also must appear as an internal wine. Then by (ia), (ii$c$), (iii) and proposition 7, the degree of divergence of the diagram is no worse than $D-2-2\min(\tilde{r}, \Delta r)$, and thus by (9.2) and (9.3) the diagram is finite.

If $\hat{S}_C$ appears, it must do so as a k-carrying lobe by corollary 5(b). Splitting the diagram into tadpoles, we see from proposition 7 and (iii) above, that the diagram is finite by (9.3).

Finally, if $\hat{S}_B$ appears, it must do so as a k-carrying lobe by corollary 5(b). If an internal $\{c'\}$ also appears in the loop then either it or another $\{c'\}$ may incorporated in a tadpole with the $\hat{S}_B$, in which case the diagram is finite by proposition 7, (ii$a$) and (9.2), or else the internal $\{c'\}$s must only lie between adjacent k-carrying $\hat{S}_A$s. In this case the $\hat{S}_B$ can be incorporated in a tadpole with $\{K-c'\}_{\mathbf{A}}$, there being no other way to attach $\hat{S}_B$ to the k-carrying $\hat{S}_A$s, as follows by corollary 5 and proposition 6. But we have already shown that such a diagram is finite. □

By proposition 8, and corollary 5 and proposition 6, we are left only to prove the convergence of diagrams either (a) consisting only of $\hat{S}_A$ and $\{c'\}$, or (b) where all the k-carrying wines are $\{L\}$s attaching to $\hat{S}_B$s, either directly to the $\mathcal{B}$s, or as in figure 17. We complete the proof of theorem 1, by showing that both these types vanish by the supertrace mechanism. As we will see, this is obvious for case (a). In case (b) the supertrace mechanism has been obscured by spontaneous symmetry breaking, the unitary gauge, and the 'missing' σ_3 described below (2.24).

Proposition 9. *Diagrams of type (a) or (b) above, vanish identically.*

In case (a) the attachments are all made with full supermatrix differentials. There are thus no σ_3s generated and the internal Wilson loop in figure 15 is $\mathrm{str}\,1 = 0$.

In constructing the diagram in case (b), if an $\{L\}$ is attached to a full $\mathcal{B}$ at tree level (i.e. before the loop is closed) then the attachment is of the form

$$\mathrm{str}\int d^D w \mathbf{X}\left(\frac{\delta}{\delta \mathbf{C}} + \nabla_\mu \cdot \frac{\delta}{\delta \mathcal{A}_\mu}\right) \mathrm{str}\int d^D x \left(\mathbf{B}_\nu + \nabla_\nu \cdot \mathbf{C}\right) \mathbf{Y}_\nu \,, \tag{9.8}$$

where we have written out l by (2.27) and the $\mathcal{B}_\nu$ it attacks, using (2.24). Here $\mathbf{X}$ and $\mathbf{Y}_\nu$ stand for the rest of the trees on either side. By proposition 6, we are interested only in the case where the $\mathcal{A}$ differential attacks the lone $\mathbf{B}_\nu$. Thus using (2.9) and (2.11), the part of the above we are interested in, evaluates to

$$\begin{aligned} -\,\mathrm{str}\int d^D x \left(\nabla_\nu \cdot \mathbf{X}\, \mathrm{d}_- \mathbf{Y}_\nu + \mathrm{d}_+ \mathbf{X}\, \nabla_\nu \cdot \mathbf{Y}_\nu\right) = \\ = \frac{1}{2}\,\mathrm{str}\int d^D x \left(\nabla_\nu \cdot \mathbf{X}\sigma_3 \mathbf{Y}_\nu \sigma_3 - \sigma_3 \mathbf{X} \sigma_3 \nabla_\nu \cdot \mathbf{Y}_\nu\right). \end{aligned} \tag{9.9}$$

We see that we have exactly one σ_3 trapped on each side of the join as in the second term of figure 2. As in figure 2, the same local process takes place on supersplitting, thus if the $\{L\}$ attachment to $\mathcal{B}$ closes the loop, a closely similar calculation shows that exactly one σ_3 gets trapped in each Wilson loop.

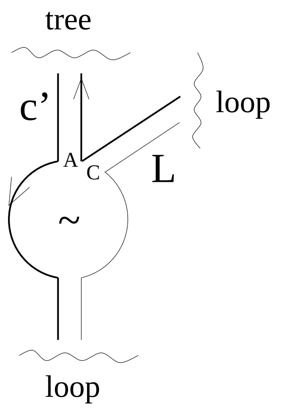


Figure 21: The partner diagram for figure 17.

For every occurence of figure 17, there is a corresponding diagram with attachments as in figure 21, where $\{c'\}$ *first* attaches to the covariant derivative in $\mathcal{B}_\nu$, i.e. to the term $i\,\mathbf{C}\mathcal{A}_\nu$ (by (2.3), (2.24) and the ordering implied in figure 21).[18]

Consider these attachments made at tree level, with $\{L\}$ and everything it attaches to at its other end being $\mathbf{X}(w)$, and the rest of $\hat{S}_B$ and its attachments being $\mathbf{Y}_\nu(x)$, as in (9.8), and $\{c'\}$ and all it attaches to being $\mathbf{Z}_\mu$. By the description above lemma 3, figure 17 is given by $-i\,\mathrm{str}\int d^D x\, \mathbf{X}\mathbf{Z}_\nu\, \mathrm{d}_- \mathbf{Y}_\nu$. On the other hand by the description directly above, figure 21 is given by $i\,\mathrm{str}\int d^D x\, \mathbf{X}\, \mathrm{d}_+ \left(\mathbf{Z}_\nu \mathbf{Y}_\nu\right)$. Recognizing that, by corollary 5, for BC bereft diagrams $\mathrm{d}_+ \mathbf{Z} = \mathbf{Z}$, these contributions sum to

$$i\,\mathrm{str}\int d^D x \mathbf{X}\mathbf{Z}_\nu \sigma_3 \mathbf{Y}_\nu \sigma_3 \,.$$

As in (9.9), we have exactly one σ_3 trapped on either side of the join. Also as above,

[18] Contributions figure 17 and figure 21 were discussed as fig. 24 of ref. [3].

a similar calculation for the case where figure 17 or figure 21 closes the loop, shows that exactly one σ_3 gets trapped in each Wilson loop in this case. Therefore we see that in diagrams of type (b), an even number of σ_3s appear in the inner Wilson loop (one for every corner or join), which thus again furnishes $\mathrm{str}\, 1 = 0$. $\square$

9.1 Divergent diagrams

The above concludes the proof of finiteness of BC bereft one-loop diagrams. We now show by example where the difficulties lie in extending the ultraviolet finiteness of the present 'unitary gauge' formulation beyond this class, in particular we provide examples of divergent one-loop diagrams with external **B**s, and divergent two-loop diagrams.

The real difficulties appear to arise through the restriction to **A** of $\{K - c'\}_{\mathbf{A}}$ in (2.27). Thus we readily obtain one-loop divergent diagrams with remaindered **B**s (cf. the lemma 3 and the discussion before it) for example figure 22. Note that since the attachment of L to $\{K - c'\}_{\mathbf{A}}$ can only be made via **A**, and the other end can only attach via **B** of $\mathcal{B}$ (by fermion number conservation given that from (2.27), $\{K - c'\}_{\mathbf{A}}$ attaches via **B**) there is no corresponding diagram that could cancel the divergence via the supertrace mechanism. Therefore some trapped σ_3s remain, after expanding as in figure 2. Noting (5.18) and the Feynman rules, one readily shows that figure 22 is quadratically divergent in $D = 4$ dimensions (the integrand $\sim 1/k^2$ for large loop momentum k).

We see that the **A** restriction destroys the supertrace mechanism as well as drifting. Another example is that of figure 23. Since the c' attachment has to be made via **A**s there is no supertrace cancellation, and from the $\hat{S}_A$ part of the RH lobe, we have again that the integrand $\sim 1/k^2$. (Note that the external **B**s are forced via the **B** differential of the $\{K - c'\}_{\mathbf{A}}$ attachment.)

The problem is particularly severe beyond one loop. It is easy to give severe cases with external **B**s, but now also BC bereft diagrams are affected, as illustrated in figure 24. Here $\{K - c'\}_{\mathbf{A}}$ is attached to $\hat{S}_A$ which is finite, but then a two-loop diagram is made by attaching L. Ls ends will drift to the joins between $\{K - c'\}_{\mathbf{A}}$ and $\hat{S}_A$ and cancel the diagrams where L attaches and drifts from the other side of

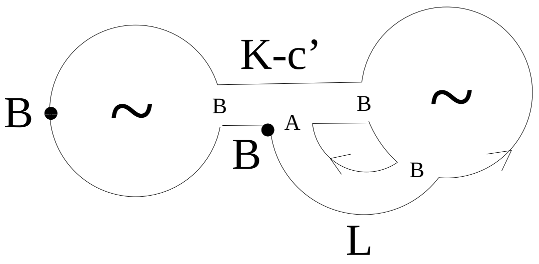


Figure 22: Divergent diagram with a remaindered **B**.

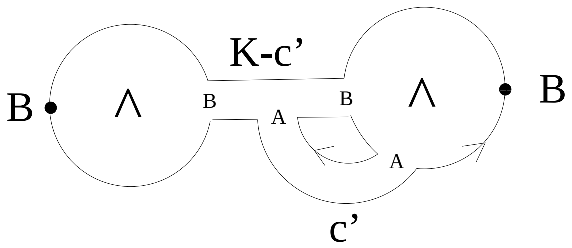


Figure 23: Divergent diagram through restriction of supertrace mechanism.

the join. However in the case that L attaches via **B** this cancellation is imperfect because the corresponding diagram does not exist for L attaching to $\{K-c'\}$. (Recall the arguments above lemma 3. Equivalently we can start by noting that when L attaches exclusively to $\hat{S}_A$ the result vanishes by the supertrace mechanism.) The loop momentum routing round L through $\hat{S}_A$ gives a violently divergent integral with an integrand that *diverges* as $k^{2\Delta r}$.

We cannot remove the **A** restriction however since the c' in figure 17 would then become K and would not then cancel against figure 21 as in proposition 9 and ref. [3], resulting in quadratically divergent one-loop diagrams (integrand $\sim 1/k^2$) even in the BC bereft sector [3].

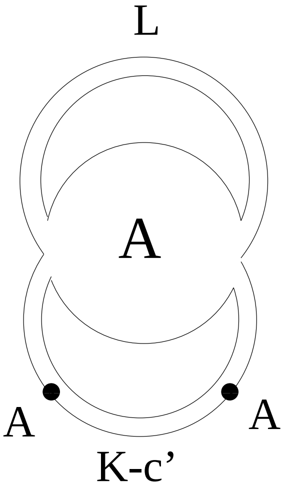


Figure 24: Divergent BC bereft two-loop diagram.

Also if we removed the **A** restriction, $\{K-c'\}$ would bite its own tail as in figure 3, and since it does so from (2.27) via the partial supermatrix **B**, there is no supertrace cancellation in the inner Wilson loop, leading by (5.19), to an unregulated integrand $\sim 1/k^2$. (Here we note that the leading divergence would superficially arise from (5.18), but this vanishes under Lorentz invariance of the k integral.) However this problem might need a separate solution as we explain below.

Actually, tail-biting is a generic problem beyond one loop or for finite N. Recall that $\{c'\}$ did not bite its own tail because the inner Wilson loop in figure 3 had to be field free in the large-N limit, thus giving $\operatorname{str} 1 = 0$. But at finite N, $\mathcal{A}$s can appear on the inner loop and thus $\operatorname{str} 1$ does not arise. Such a term could still be logarithmically divergent in $D = 4$ dimensions (integrand $\sim 1/k^4$).

At two-loops the subtleties of the large-N limit of the flow equation itself [1] may allow such divergent tail biting diagrams even at $N = \infty$ as illustrated in figure 25, providing neither field free Wilson loop vanishes by the supertrace mechanism.

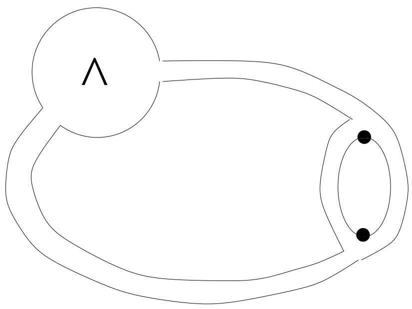


Figure 25: An $N = \infty$ divergent two-loop tail-biting diagram.

For these reasons, extending these ideas to finite N and beyond one-loop seem to require returning to more geometric covariantization (e.g. straight line wines) and regulating the wines as already described in [3, 1] so as to eliminate diagrams of form figure 3.

As we have seen, apart from this, divergences appear to arise only because of the restriction to **A** in $\{K-c'\}_{\mathbf{A}}$ which is necessary to ensure that the hidden supertrace mechanism described in proposition 9 works at least in BC bereft one-loop diagrams. We expect that in a finite manifestly local SU$(N|N)$ approach such problems will be absent.

10. Conclusions

We have presented a methodology which for the first time allows manifestly gauge invariant continuum calculations to be performed. No gauge fixing or ghosts are required, and the full power and beauty of local gauge invariance is directly incorporated. We have demonstrated just a few of the consequences: the quantum gauge field does not renormalize — only the gauge coupling requires renormalization; possible counterterms are strongly constrained, easily and elegantly determined; the one-loop β function falls out in a manifestly gauge invariant manner; the gauge invariant continuum wilsonian effective action is for the first time precisely defined.

It is important to recognize that the crucial problem solved in constructing a successful manifestly gauge invariant calculus in quantum field theory, is the passage to the continuum limit, equivalently renormalization. This is achieved by at the same time solving the long standing problem of combining gauge invariance with the exact RG. (In the exact RG, the continuum limit is almost automatic; in the gauge sector we only need to require a finite coupling constant $g(\Lambda)$ [7].)

However, in order to write down such a gauge invariant exact RG, one must first solve the problem of finding a continuum gauge invariant physical cutoff. As we have seen from this paper and ref. [3], this is in a sense uniquely given by spontaneously broken SU($N|N$) gauge theory with covariant higher derivatives. This method of regularization stands separately and is interesting and useful in its own right. It will be discussed more fully in ref. [11]. We also comment further below.

Our method should be compared with previous approaches to the exact RG in gauge theory which implement a physical cutoff at the expense of not only fixing, but breaking, the gauge invariance at least at intermediate stages [17], and it should be compared from this point of view with stochastic quantization (the manifestly gauge invariant form of which fails to regulate in the gauge orbit directions) [18] and the famous loop-space, a.k.a. Migdal-Makeenko or Dyson-Schwinger, approach [19, 20, 21]– [22] as already indicated in refs. [3, 1], where it is unknown how to formulate it at other than the bare level [21]. At finite N, Mandelstam's relations [23, 20], which encode the overcomplete nature of the Wilson loop representation, also significantly complicate this approach since both equations are relations between intersecting loops.

Here this overcounting has no great consequence, amounting to a harmless overparametrization of the effective action [3]; intersecting loops do not here carry any special significance. We stress that while it is possible and we believe useful to express the pure gauge sector and pure gauge sections geometrically in terms of Wilson loops and lines respectively, it is not necessary for our formulation. Nor need the apparent non-locality be any more than usual. Indeed if we choose c^{-1} and $\tilde{c}^{-1}$ to be polynomials, then the seed action $\hat{S}$ is just a sum over a finite number of ultralocal operators (i.e. vertices polynomial in their momenta) as would be normal for a bare action.[19] The effective action S is necessarily only quasilocal (i.e. Taylor expandable in momenta) as is the case for any wilsonian effective action.

We should emphasise some subtleties in our construction. Even the inclusion of Pauli-Villars fields as part of the effective cutoff in an exact RG is novel, and has to be done carefully so that flow to lower energies amounts to *lowering* their masses (rather than raising them as naturally occurs for a relevant perturbation such as a mass) and thus corresponds to integrating out. We saw that at the *classical level* the solutions involve integrals over the effective cutoff Λ which diverge, requiring integration constant 'counterterms' or in effect a finiteness prescription for these Λ integrals.

The Pauli-Villars contributions lead to finiteness in the momentum integrals, by subtracting separately divergent contributions. The answers are finite — but well defined only after applying and removing a preregulator. (In our case we took $D < 4$ and only let $D \to 4$ at the end of the calculation.) Only the symmetries respected in this procedure are guaranteed preserved. It is important in a regularisation such as this, that appears to generalise so straightforwardly to in particular anomalous gauge theories, that there are subtleties in its construction. For non-anomalous theories, an imperfect pre-regulator can be repaired by adding symmetry-breaking operators that vanish on removing the pre-regulator, but for anomalous theories this procedure will fail.

From below (2.24), we also recall the wrong sign action for A^2_μ. As we noted below (2.2), and in ref. [3], this unphysical gauge field decouples from the physical gauge field A^1_μ at energies much less than Λ, but problems could potentially arise for A^1 if only at the non-perturbative level. At first sight the wrong-sign A^2 action leads to instability, however the fact that the kinetic term has also the wrong sign results instead in unitarity violations, whose effect in the physical sector should disappear in the limit that the overall cutoff is removed [11].

We now understand the wrong sign A^2 action as an inevitable consequence of the underlying $\mathrm{SU}(N|N)$ gauge theory. The idea of regularising via such a gauge theory is another novel development in this paper. (For some previous applications of $\mathrm{SU}(N|N)$ gauge theory see refs.[20] [24].) Note that geometrically this corresponds to realising a superspace in the fibres of the principal bundle rather than in the base space as would correspond to the usual spacetime supersymmetry.

As we saw in section 6.9, an intriguing property, in fact inherent to $\mathrm{SU}(N|N)$, is a duality that exchanges the rôle of A^1 and A^2 and thus also in a sense changes g^2 to $-g^2$. The present 'unitary gauge formulation' closely imitates a spontaneously broken $\mathrm{SU}(N|N)$ theory but actually differs in some small details (as discussed in

[19] Despite some similarities $\hat{S}$ is a renormalized action however.

[20] The author thanks John Tighe for bringing these and ref. [14] to his attention.

sections 2 and 6.6), and it is not clear to us that this duality survives spontaneous breaking in a true local SU($N|N$) invariant formulation. We must also be mindful that the present formulation only regulates to one loop with any number of external gauge fields. Nevertheless, it is intriguing to note that the duality symmetry, the local SU($N|N$) and the global fermionic U(1), do not commute with each other. In view of the intimate connection of g^2 with σ_3 as suggested by duality, which may be taken to imply that σ_3 appears with the same power as g^2, it seems that we should regard the coupling constant as the space-time independent field $g^2\sigma_3$. The action of local SU($N|N$) then forces us to consider it a space-time dependent field. Perhaps these ideas hold the key to develop this duality at a deeper level.

We saw however that the duality is at least obscured when we come to renormalize. To renormalize the A^2 sector we should introduce a separate coupling g_2 for A^2. (We expect this to be true also in the complete SU($N|N$) theory.) A very intriguing scenario now arises as is clear from (6.11). The wrong sign action for A^2 is equivalent to the wrong sign g_2^2, and means that for negative β_1, g_2 is not asymptotically free but trivial. As $\Lambda_0 \to \infty$, the continuum limit for the physical Yang-Mills field A^1 is reached by $g_0 \to 0$. But in this limit g_2 vanishes, and thus it would appear that for dynamical reasons the unphysical A_2 sector entirely decouples and loses all interactions in the continuum limit.

Finally, let us remark that the formalism presented here appears to open many doors, and suggests many further avenues of exploration. We note that there is considerable freedom: the use of completeness relations for the generators, coincident lines, power law cutoff functions, or the same covariantization for the different kernels, is not necessary. Other interactions in $\hat{S}$ are possible, and other ways of arranging the covariantization and regularisation may be considered along similar lines [25]. This may help to understand more deeply holographic RG flow in the AdS/CFT correspondence and string theory [26]. We have already touched on the issues of extending the regularisation to finite N [11], of separately developing the SU($N|N$) regularisation [11], of formulating a manifestly SU($N|N$) invariant exact RG and generalisation to other groups. We do not directly compute correlators of gauge fields, the integration over modes being indirect through in effect iterating infinitessimal gauge invariant changes of variables [3]. Nevertheless we expect that gauge invariant correlators can be computed by introducing sources for the appropriate gauge invariant operators and absorbing these sources as space-time dependent couplings. As $\Lambda \to 0$, all modes are integrated out and the partition function $\mathcal{Z}$ should then be just proportional to e^{-S}. Differentiating with respect to the sources should then allow all the physics to be extracted. We should also understand if/how gauge *variant* operators are gauge averaged over. Other future directions include higher loop calculations such as β_2, investigating the Gribov problem in the continuum — perhaps through a limit as gauge fixing is removed, incorporation of matter fields in other representations; U(1) gauge theory e.g. QED, through U(1|1), should be much simpler in many respects since the gauge field kernels do not need covariantizing [3, 27]. We have already touched on anomalies in this framework but these deserve further investigation. This formalism should allow investigations of instantons, renormalons and other controlled non-perturbative effects in a manifestly gauge invariant way. It looks possible to generalise this formalism to local coordinate invariance and thus a non-perturbative continuum framework for quantum gravity and supergravity. Of course the exact RG is tailor made to investigate Seiberg-Witten methods [4] and super Yang-Mills theory more generally at a deeper level. And last but by no means least, we are hopeful that fully non-perturbative approximation methods can be developed to allow accurate analytic continuum calculations in realistic gauge theories, e.g. SU(3) and QCD. Such approximation methods can involve for example the large-N (colour) limit [1, 3]. Many other potentially powerful approximation methods are also available within the exact RG framework [3, 7, 28, 29].

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